Mathematical

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THE

MATHEMATICAL

REPOSITORY.

BY T. LEYBOURN,

)F THE ROYAL MILITARY COLLEGE.

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THE

MATHEMATICAL REPOSITORY.

ARTICLE I.

An Investigation of Theorems relating to the sums of the Powers of perpendiculars to all the sides of a regular polygon, from any point in the circumference of its inscribed circle.

By *Mr.* JAMES CUNLIFFE, *Bolton-le-Moors.*

ARTICLE 1. LEMMA. To find the sums of the various powers of the roots of the equation

$v^n - Pv^{n-1} + Qv^{n-2} - Rv^{n-3} + Sv^{n-4}$ &c. $= 0$; n being a whole number.

Let the roots of the equation be denoted by a, b, c, d &c. the number of which is n, as is very well known: Then by the property of equations

$$v^n - Pv^{n-1} + Qv^{n-2} - Rv^{n-3} + Sv^{n-4} \text{ \&c.} = (v-a)(v-b)(v-c)(v-d) \text{ \&c. to } n \text{ factors.}$$

By taking the fluxions of the hyperbolic logarithms of each side of the equation, and dividing by $\dot{v}$ there will be had

$$\frac{nv^{n-1} - (n-1)Pv^{n-2} + (n-2)Qv^{n-3} - (n-3)Rv^{n-4} + (n-4)Sv^{n-5} \text{ \&c.}}{v^n - Pv^{n-1} + Qv^{n-2} - Rv^{n-3} + Sv^{n-4} \text{ \&c.}}$$

$$=\left\{\begin{array}{l}\frac{1}{v-a}+\frac{1}{v-b}+\frac{1}{v-c}+\frac{1}{v-d}\&c.\\ \text{to } n \text{ terms}\end{array}\right\}=\left\{\begin{array}{l}\frac{1}{v}+\frac{a}{v^2}+\frac{a^2}{v^3}+\frac{a^3}{v^4}+\frac{a^4}{v^5}\ \&c.\\ \frac{1}{v}+\frac{b}{v^2}+\frac{b^2}{v^3}+\frac{b^3}{v^4}+\frac{b^4}{v^5}\ \&c.\\ \frac{1}{v}+\frac{c}{v^2}+\frac{c^2}{v^3}+\frac{c^3}{v^4}+\frac{c^4}{v^5}\ \&c.\\ \frac{1}{v}+\frac{d}{v^2}+\frac{d^2}{v^3}+\frac{d^3}{v^4}+\frac{d^4}{v^5}\ \&c.\end{array}\right.$$

by actually expanding the fractions $\frac{1}{v-a}$, $\frac{1}{v-b}$ &c. where it is evident that the vertical columns must be continued to n terms.

Put $A = a + b + c + d$ &c. to n terms
$B = a^2 + b^2 + c^2 + d^2$ &c. to n terms
$C = a^3 + b^3 + c^3 + d^3$ &c. to n terms
$D = a^4 + b^4 + c^4 + d^4$ &c. to n terms. Then will

$$\frac{nv^{n-1}-(n-1)Pv^{n-2}+(n-2)Qv^{n-3}-(n-3)Rv^{n-4}+(n-4)Sv^{n-5}\ \&c.}{v^n-Pv^{n-1}+Qv^{n-2}-Rv^{n-3}+Sv^{n-4}\ \&c.}$$

$$=\frac{n}{v}+\frac{A}{v^2}+\frac{B}{v^3}+\frac{C}{v^4}+\frac{D}{v^5}\ \&c.$$ and by multiplying by

$v^n - Pv^{n-1} + Qv^{n-2} - Rv^{n-3} + Sv^{n-4}$ &c. there will be had

$$nv^{n-1}-(n-1)Pv^{n-2}+(n-2)Qv^{n-3}-(n-3)Rv^{n-4}+(n-4)Sv^{n-5}\ \&c.$$
$$\begin{aligned}= nv^{n-1}&+Av^{n-2}+Bv^{n-3}+Cv^{n-4}+Dv^{n-5}\ \&c.\\ &-nPv^{n-2}-PAv^{n-3}-PBv^{n-4}-PCv^{n-5}\ \&c.\\ &\qquad+nQv^{n-3}+QAv^{n-4}+QBv^{n-5}\ \&c.\\ &\qquad\qquad-nRv^{n-4}-RAv^{n-5}\ \&c.\\ &\qquad\qquad\qquad+nSv^{n-5}\ \&c.\end{aligned}$$

Equating the co-efficients of the homologus terms, there will be had, $-(n-1)P = A - nP$; $(n-2)Q = B - PA + nQ$; $-(n-3)R = C - PB + QA - nR$; $(n-4)S = D - PC + QB - RA + nS$, &c. From the first of these equations [...] rom $B = PA - 2Q = P^2 - 2Q$; from the [...] $3R = P^3 - 3PQ + 3R$; and from the [...] $+ RA - 4S = P^4 - 4P^2Q + 2Q^2 +$

It

It may be remarked, by the way, from what is just deduced, that if, in any equation, the indices of the unknown quantity are whole numbers, and the co-efficients of the several terms rational; then the sums of the various powers of the roots will be rational also. The truth of this remark is evident from barely inspecting the formulæ themselves; for nothing is to be found in the formulæ but rational quantities. Therefore, if imaginary, or surd quantities are any ways concerned in the roots themselves, the affirmative and negative parts of such imaginary or surd quantities, which are concerned in the sums of the various powers of the roots must mutually destroy each other.

Art. 2. If v be the versed sine of a circular arc A, whose radius is r; and V the versed sine of any multiple n of that arc or of nA; then per Cor. 2, Prop. 29, Book I. of Emerson's Trigonometry,

$$\frac{(2v)^n}{2r^{n-1}} - \frac{2nrA}{2v} - \frac{(2n-3)rB}{2.2v} - \frac{(2n-4)(2n-5)rC}{3(2n-3).2v} - \frac{(2n-6)(2n-7)rD}{4(2n-4).2v}$$

&c. $= \pm$ V; where A, B, C, D &c. denote the preceding terms with their sines: therefore by restoring the values of A, B, C, D, &c. and multiplying the resulting expression by $\frac{r^{n-1}}{2^{n-1}}$ it becomes

$$\left.\begin{array}{l} v^n - nrv^{n-1} + \dfrac{n(2n-3)r^2v^{n-2}}{2^2} - \dfrac{n(2n-4)(2n-5)r^3v^{n-3}}{2^3.3} \\ \qquad + \dfrac{n(2n-5)(2n-6)(2n-7)r^4v^{n-4}}{2^4\cdot 3\cdot 4}\ \&c. \end{array}\right\} =$$

$$\frac{r^{n-1}}{2^{n-1}} \times \pm V.$$

The former of the preceding equations will also express the versed sine of the arcs C + nA, 2C + nA, 3C + nA, 4C + nA &c. where C denotes the whole periphery of the circle. Therefore the several values of v, or roots of the preceding equation will be the versed sines of the arcs, A, $\frac{C}{n}$ + A, $\frac{2C}{n}$ + A, $\frac{3C}{n}$ + A, $\frac{4C}{n}$ + A &c. continued to n terms.

Art. 3. Fig. 496, pl. 27. If a regular polygon the number of whose sides is n, be circumscribed about a circle; M′, M″,

M‴,.......M^n being the points of contact with the sides; and if from any point P, in the arc $M'M^n$ perpendiculars PQ, PR, PS, &c. be drawn to the sides M′, M″, M‴ &c. these ⊥s will be respectively equal to the versed sines of the arcs PM′, PM″, PM‴ &c.

Demonstration. Through the point of contact M′, draw the diameter M′N and Pn perpendicular to it. Then QM′n is a right angle (Eu. 18. III), wherefore the figure QM′nP is a rectangle, and (Eu. 34. I.), PQ = M′n, the versed sine of the arc PM′. And in the same manner it may be demonstrated that PR is equal to the versed sine of the arc PM″ &c.

Now if the diameter M′N = $2r$, C = the whole periphery of the circle, and arc PM′ = A; then will $\frac{C}{n}$ be equal to each of the arcs M^nM', M′M″, M″M‴ &c. therefore PM″ $= \frac{C}{n} + A$, PM‴ $= \frac{2C}{n} + A$, &c. and the perpendiculars PQ, PR, PS, &c. being respectively equal to the versed sines of the arcs A, $\frac{C}{n} + A$, $\frac{2C}{n} + A$ &c. will manifestly be the roots of the equation in Art. 2.

Art 4. It is required to find the sums of the various powers of the perpendiculars PQ, PR, PS, &c.

By comparing the co-efficients of the homologous terms of the equations in Article 1 and 2, there will be had $P = nr$; $Q = \frac{n(2n-3)r^2}{2^2}$; $R = \frac{n(2n-4)(2n-5)r^3}{2^3 \cdot 3}$; $S = \frac{n(2n-5)(2n-6)(2n-7)r^4}{2^4 \cdot 3 \cdot 4}$ &c.

Then $A = P = nr =$ to the sum of all the perpendiculars PQ, PR, PS, &c.

$B = PA - 2Q = n^2r^2 - \frac{n(2n-3)r^2}{2} = \frac{3nr^2}{2} =$ to the sum of the squares of all the perpendiculars PQ, PR, PS, &c.

$C = PB - QA + 3R = \frac{3n^2r^3}{2} - \frac{n^2(2n-3)r^3}{2^2} + \frac{n(2n-4)(2n-5)r^3}{2^3} = \frac{5nr^3}{2} =$ to the sum of the cubes of all the perpendiculars PQ, PR, PS, &c.

D=

$$D = PC - QB + RA - 4S = \frac{5n^2r^4}{2} - \frac{3n^2(2n-3)r^4}{2^3}$$
$$+ \frac{n^2(2n-4)(2n-5)r^4}{2^3 \cdot 3} - \frac{n(2n-5)(2n-6)(2n-7)r^4}{2^4 \cdot 3}$$
$= \frac{35}{8} r^4 =$ to the sum of the fourth powers of all the perpendiculars PQ, PR, &c.

And by proceeding in the same way we shall obtain $\frac{63nr^5}{8}$ for the sum of the fifth powers of the said perpendiculars.

The two formulæ for the sum of the squares and sum of the cubes of the perpendiculars PQ, PR, PS, &c. agree perfectly with what is demonstrated by Messrs. Lowry and Swale in the first volume of the Repository.

ARTICLE II.

Improved Solutions to some Curious Mathematical Problems.

PROB. I. *By Mr.* JAMES GUNLIFFE.

TO find three square numbers, such, that the difference of every two of them may be a square number.

SOLUTION.

Let the numbers be denoted by x^2, y^2, and z^2; then by the question $x^2-y^2=a^2$, $x^2-z^2=b^2$, and $y^2-z^2=$ a square. From the two first equations $x^2=a^2+y^2=b^2+z^2$.

Put $b=a+sv$, and $z=y=rv$. Then
$a^2+y^2=b^2+z^2=(a+sv)^2+(y-rv)^2=a^2+y^2+2sav-2ryv+s^2v^2+r^2v^2$; therefore,

$$v=\frac{2ry-2sa}{r^2+s^2}, \text{ and } z=y-rv=y-\frac{2r^2y-2rsa}{r^2+s^2}=\frac{2rsa-y(r^2-s^2)}{r^2+s^2}.$$

But $x^2=a^2+y^2$, theref. put $(r^2+s^2)2mn=a$, and $(r^2+s^2)(m^2-n^2)=y$, then will
$x=(r^2+s^2)(m^2+n^2)$, and $z=4rsmn-(r^2-s^2)(m^2-n^2)$. Wherefore

 y^2-z^2

$$y^2-z^2=(r^2+s^2)^2(m^2-n^2)^2-(r^2-s^2)^2(m^2-n^2)^2+8rs(r^2-s^2)mn(m^2-n^2)-16r^2s^2m^2n^2$$

$$=4r^2s^2(m^2-n^2)^2+8rs(r^2-s^2)mn(m^2-n^2)-16r^2s^2m^2n^2=$$ a square by the question. Assume $2rs(m^2+n^2)-2mn(r^2-s^2)$ for its root, and then,

$$4r^2s^2(m^2-n^2)^2+8rs(r^2-s^2)mn(m^2-n^2)-16r^2s^2m^2n^2=[2rs(m^2+n^2)-2mn(r^2-s^2)]^2$$

$$=4r^2s^2(m^2+n^2)^2-8rs(r^2-s^2)mn(m^2+n^2)+(r^2-s^2)^2 4m^2n^2,$$ which being properly reduced gives $\frac{m}{n}=\frac{r^4+6r^2s^2+s^4}{4rs(r^2-s^2)}$; whence it appears that m and n may be expounded by any two numbers in the ratio of $r^4+6r^2s^2+s^4$ to $4rs(r^2-s^2)$ respectively, where r and s may be taken at pleasure.

For example, if $r=3$ and $s=1$, then $\frac{m}{n}=\frac{136}{96}=\frac{17}{12}$; therefore put $m=17$, and then $n=12$, whence $x=4330$, $y=1450$, and $z=1288$, which are the roots of three numbers with the required property. Or their halves, viz. 2165, 725 and 644 will be the roots of three squares that will answer.

Or the solution may be as follows.

Making use of the expressions for x, y, and z, which were obtained in the foregoing solution, it is plain that if we can determine $\frac{z+y}{2}$ and $\frac{z-y}{2}$, each equal to a rational square, the question will be answered, for then $\frac{z+y}{2}\times\frac{z-y}{2}=\frac{z^2-y^2}{4}$ must be a rational square, because each of its factors is so. Now $\frac{z+y}{2}=s^2(m^2-n^2)+2rsmn$, and $\frac{z-y}{2}=r^2(n^2-m^2)+2rsmn$ are both to be squares; let the first of them be divided by s^2, and the other by r^2, and the quotients will be m^2-n^2+2tmn, and $n^2-m^2+\frac{2mn}{t}$, by putting $t=\frac{r}{s}$. Now these two last expressions must manifestly be still square numbers; therefore assume $qn-m$ for the root of the first of them, that is, put $m^2-n^2+2tmn=(qn-m)^2=q^2n^2-2qnm+m^2$, whence $n=2m\times\frac{t+q}{q^2+1}$, which being written for n in the other expression there w...

n^2-m^2

$n^2 - m^2 + (2nm \div t) = 4m^2(t+q)^2 \div (q^2+1)^2 - m^2 + 4m^2(t+q) \div t(q^2+1) =$
$m^2 \div t^2(q^2+1)^2 \times [4t^2(t+q)^2 - t^2(q^2+1)^2 + 4t(t+q)(q^2+1)] =$ a square, and therefore,
$4t^2(t+q)^2 - t^2(q^2+1)^2 + 4t(t+q)(q^2+1) = 4t^4 + 8t^3q + t^2(6q^2 - q^4 + 3) + 4tq(q^2+1)$ must be a square. Assume $2t^2 + 2tq$ for the root of this square, and then
$4t^4 + 8t^3q + t^2(6q^2 - q^4 + 3) + 4tq(q^2+1) = (2t^2 + 2tq)^2 = 4t^4 + 8t^3q + 4t^2q^2$, whence

$t = \frac{r}{s} = \frac{4q(q^2+1)}{q^4 - 2q^2 - 3}$, and by means of this value of t,

$n = 2m \times \frac{t+q}{q^2+1} = m \times \frac{2q(q^2+1)}{q^4 - 2q^2 - 3}$, or $\frac{m}{n} = \frac{q^4 - 2q^2 - 3}{2q(q^2+1)}$.

From these conclusions it appears that r and s may be expounded by any two numbers in the ratio of $4q(q^2+1)$ to $q^4 - 2q^2 - 3$, and at the same time m and n must be taken in the ratio of $q^4 - 2q^2 - 3$ to $2q(q^2+1)$ respectively where q may be taken at pleasure.

Example 1. If $q = 2$, then $\frac{r}{s} = \frac{4q(q^2+1)}{q^4 - 2q^2 - 3} = \frac{40}{5} = \frac{8}{1}$, and $\frac{m}{n} = \frac{q^4 - 2q^2 - 3}{2q(q^2+1)} = \frac{5}{20} = \frac{1}{4}$; wherefore we may put $r = 8$, $s = 1$, $m = 1$, and $n = 4$; whence $x = (r^2 + s^2)(m^2 + n^2) = 65 \times 17 = 1105$, $y = (r^2 + s^2)(m^2 - n^2) = 65 \times - 15 = 975$, and $z = 4rsmn - (r^2 - s^2)(m^2 - n^2) = 128 + 945 = 1073$ which are the roots of three squares that will answer, and differ materially from those found by the first method of solution.

Example 2. Suppose $q = 4$, then $\frac{r}{s} = \frac{16 \times 17}{221} = \frac{16}{13}$, and $\frac{m}{n} = \frac{221}{8 \times 17} = \frac{13}{8}$, therefore we may take $r = 16$, $s = 13$, $m = 13$, and $n = 8$, then $x = 425 \times 233 = 99025$, $y = 425 \times 105 = 44625$, and $z = 86528 - 9135 = 77393$ being the roots of three other squares that will answer the question.

Example 3. If $q = 5$, then $\frac{r}{s} = \frac{20 \times 26}{572} = \frac{10}{11}$, and $\frac{m}{n} = \frac{572}{10 \times 26} = \frac{11}{5}$, therefore we may take $r = 10$, $s = 11$, $m = 11$, and $n = 5$. Hence $x = 221 \times 146 = 32266$, $y = 221 \times 96 =$ 21216

[illegible], and $z = 24300 + 11[illegible] = [illegible]$, which [illegible] [illegible] will answer [illegible] [illegible] and [illegible] will be the roots of [illegible]

[illegible] value of t may be obtained as follows.

[illegible] $q^4 + 4q^3 + [illegible] - q^2 + [illegible] + [illegible](q^2+1)$ [illegible] Sume $2q^2 + 2q - \frac{q^4 - 2q^2 + 3}{4}$ [illegible], and then $q^4 + 2q^3 + [illegible] 6q^2 - [illegible] + [illegible](q^2+1) =$ $[illegible] q^2 + 2q - \frac{q^4 - 2q^2 - 3}{4})^2 = 4q^4 + 8q^3 + [illegible](5q^2 - q^4 + 3) - q$

$$(q^4 - 2q^2 - 3) + \left(\frac{q^4-2q^2-3}{16}\right)^2,$$

which being properly reduced gives $t = \frac{r}{s} = \frac{(q^4 - 2q^2 - 3)^2}{16q(q^2 + 1)^2}$ in which q may be taken at pleasure.

If $q = 1$, then $t = \frac{r}{s} = \frac{16}{16 \times 4} = \frac{1}{4}$; whence $n = 2m \times \frac{t + q}{q^2 + 1} = \frac{5m}{4}$, or $\frac{m}{n} = \frac{4}{5}$; hence it appears, that we may take $r = 1$, $s = 4$, $m = 4$, and $n = 5$; whence $x = (r^2 + s^2)(m^2 + n^2) = 17 \times 41 = 697$, $y = (r^2 + s^2)(m^2 - n^2) = 17 \times -9 = -153$, and $z = 4rsmn - (r^2 - s^2)(m^2 - n^2) = 320 - 135 = 185$, which are the roots of three squares that will answer, and are properly the roots of the least numbers that have the required property.

Or the solution may be as follows.

Let x^2, y^2, and z^2 denote the required squares, then by the question $x^2 - y^2 = a^2$, and $x^2 - z^2 = b^2$, the difference of these two equations is $y^2 - z^2 = b^2 - a^2$, which must also be a square by the question: put $y = b - rv$, and $z = a - sv$, then $y^2 - z^2 = b^2 - a^2 = (b - rv)^2 - (a - sv)^2 = b^2 - a^2 - 2rbv + 2sav + r^2v^2 - s^2v^2$, which gives

$v = \frac{2rb - 2sa}{r^2 - s^2}$; whence $y = b - rv = \frac{2rsa - b(r^2 + s^2)}{r^2 - s^2}$, & $z =$

$$a - sv = \frac{a(r^2 + s^2) - 2rsb}{r^2 - s^2}$$

Now $y^2 - z^2 = b^2 - a^2$ must be a square number by the question, therefore put $(s^2 - r^2)(m^2 - n^2) = b$, and $(s^2 - r^2)\, 2mn = a$, then,

$y=(r^2+s^2)(m^2+n^2)-4rsmn$, and $z=2rs(m^2+n^2)-2mn(r^2+s^2)$; from whence and from the first equation,

$t^2=x^2+y^2=(s^2-r^2)^2 4m^2n^2+(r^2+s^2)^2(m^2+n^2)-8rs(r^2+s^2)mn(m^2+n^2)+16r^2s^2m^2n^2$

$=(r^2+s^2)^2(m^2+n^2)-8rs(r^2+s^2)mn(m^2+n^2)+(r^2+s^2)^2 4m^2n^2=$ a square.

Assume $x=4rsmn-(r^2+s^2)(m^2-n^2)$, then will

$t^2=(r^2+s^2)^2(m^2+n^2)^2-8rs(r^2+s^2)mn(m^2+n^2)+(r^2+s^2)4m^2n^2$

$=[4rsmn-(r^2+s^2)(m^2-n^2)]^2=16r^2s^2m^2n^2-8rs(r^2+s^2)mn(m^2-n^2)+(r^2+s^2)^2(m^2-n^2)^2$,

which by proper reduction gives $m=n\times\frac{2rs(r^2+s^2)}{r^4+s^4}$; put $n=r^4+s^4$; then m $2rs(r^2+s^2)$, where r and s may be taken at pleasure.

Example. If $r=2$ and $s=1$, then $n=r^4+s^4=17$, and $m=2rs(r^2+s^2)=20$; whence $x=4rsmn-(r^2+s^2)(m^2-n^2)=2165$, $y=(r^2+s^2)(m^2+n^2)-4rsmn=725$, and $z=2rs(m^2+n^2)-2mn(r^2+s^2)=-644$.

Prob. II. *By Mr.* James Cunliffe.

To find three square numbers, such, that the sum of every two of them may be a square number.

Solution.

Let x^2, y^2, and z^2 denote the three squares:—then by the question $x^2+y^2=a^2$, $x^2+z^2=b^2$, and $y^2+z^2=$ a square; from the two first of these equations $x^2=a^2-y^2=b^2-z^2$. It appears, therefore, from hence, that a, and y, must be so expressed, that the difference of their squares a^2-y^2 may be a rational square, whose root is the value of x; and moreover, b and z, must be determined in such a manner from thence, that $a^2-y^2=b^2-z^2$. From what is done in the last solution to the preceding question it is obvious that we may take $(s^2-r^2)(m^2+n^2)=a$, and $(s^2-r^2)2mn=b$, whence $x=(s^2-r^2)(m^2-n^2)$; for then $z=2rs(m^2+n^2)-2mn(r^2+s^2)$. Wherefore

$y^2+z^2=(s^2-r^2)^2 4m^2n^2+4r^2s^2(m^2+n^2)^2-8rs(r^2+s^2)mn(m^2+n^2)+(r^2+s^2)^2 4m^2n^2=4r^2s^2(m^2+n^2)^2-8rs(r^2+s^2)mn(m^2+n^2)+(r^4+s^4)8m^2n^2=$ a square by the question. Assume $2rs(m^2-n^2)+(r^2+s^2)2mn$ for the root of this square; then will

$4r^2s^2(m^2+n^2)^2-8rs(r^2+s^2)mn(m^2+n^2)+(r^4+s^4)8m^2n^2=[2rs(m^2-n^2)+(r^2+s^2)2mn]^2=4r^2s^2(m^2-n^2)^2+8rs(r^2+s^2)mn(m^2-n^2)+(r^2+s^2)^2 4m^2n^2$, which by reduction gives $n=m\times\frac{4rs}{r^2+s^2}$; put $m=r^2+s^2$, and then $n=4rs$, where r and s may be taken at pleasure.

Example.

Example. If $r = 1$ and $s = 2$, then $m = r^2 + s^2 = 5$, and $n = 4rs = 8$, whence $x = (s^2 - r^2)(m^2 - n^2) = 3 \times -39 = -117$, $y = (s^2 - r^2)2mn = 3 \times 80 = 240$, and $z = 2rs(m^2 + n^2) - (r^2 + s^2)2mn = 4 \times 89 - 5 \times 80 = -44$. Therefore 240, 117 and 44 are the roots of three ſquares the ſum of every two of which will be a ſquare, and are the leaſt that have hitherto been found. The diſcovery of theſe numbers is attributed to EULER; but whoever was the firſt diſcoverer, they are obviouſly the firſt numbers that preſent themſelves by the above method of ſolution.

(To be continued.)

ARTICLE III.

ON CONTINUED SURDS, &c.

By Mr. BENJ. GOMPERTZ.

1. IF $x^2 - x = a$, x will be equal

$$\sqrt{a+x} = \sqrt{a+\sqrt{a+x}} = \sqrt{a+\sqrt{a+\sqrt{a+x}}} = \sqrt{a+\sqrt{a+\sqrt{a, \textit{ad infinitum}.}}}$$

II. If $x = \sqrt[3]{a^2b}$, x^4 will be $= a^2bx$, and therefore,

$$x = \sqrt{a+\sqrt{bx}} = \sqrt{a\sqrt{b\sqrt{a\sqrt{bx}}}} = \sqrt{a\sqrt{b\sqrt{a\sqrt{b, \textit{ad infinitum}.}}}}$$

Which demonſtrates *John Bernoulli's* Solutions, See *Emerſon's Miſcellanies, page* 433; theſe, Emerſon affirms to be expreſſions of infinite value, and on that account that Bernoulli's Solutions are wrong. That they are not of infinite value is plain from above, as alſo from the ſolutions in the Miſcellanies; and that the expreſſions are not even diverging may be thus generally demonſtrated.

1. If a be equal or greater than unity, $\sqrt{a+\sqrt{a}}$ may be proved by the binomial theorem to be leſs than $\sqrt{a} + \frac{1}{2}$, therefore $\sqrt{a+\sqrt{a+\sqrt{a}}}$ is leſs than $\sqrt{a+\frac{1}{2}+\sqrt{a}}$, and therefore

therefore by all means lefs than $\sqrt{a+\frac{1}{2}+\sqrt{a+\frac{1}{2}}}$, or its equal $\sqrt{c+\sqrt{c}}$, (putting $c = a + \frac{1}{2}$), and ftill farther lefs than $\sqrt{c}+\frac{1}{2}$, therefore $\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a}}}}$ is lefs than $\sqrt{a+\frac{1}{2}+\sqrt{c}}$, or its equal $\sqrt{c+\sqrt{c}}$, therefore far less than $\sqrt{c}+\frac{1}{2}$, and proceeding thus we continually find the fame expreffion $\sqrt{c}+\frac{1}{2}$ to be greater than $\sqrt{a+\sqrt{a+\sqrt{a+\&c.}}}$ and fo much the more fo as we proceed.

Or thus whether a be greater or lefs than unity.

Put $b = a + 1$, therefore $\sqrt{a+\sqrt{a}}$ is lefs than $\sqrt{b+\sqrt{b}}$ and ftill lefs than $\sqrt{b}+\frac{1}{2}$, (b being greater than unity); therefore $\sqrt{a+\sqrt{a+\sqrt{a}}}$ is lefs than $\sqrt{a+\frac{1}{2}+\sqrt{b}}$ and ftill lefs than $\sqrt{b+\sqrt{b}}$ and ftill lefs than $\sqrt{b}+\frac{1}{2}$; therefore $\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a}}}}$ is lefs than $\sqrt{a+\frac{1}{2}+\sqrt{b}}$ and ftill lefs than $\sqrt{b+\sqrt{b}}$ and ftill lefs than $\sqrt{b}+\frac{1}{2}$, and thus we continually get the fame expreffion $\sqrt{b}+\frac{1}{2}$ to be greater than $\sqrt{a+\sqrt{a+\sqrt{a+\&c.}}}$ and the more fo the further we proceed.

That the expreffion is greater than a needs no demonftration.

2. Here the expreffion may be written thus

$$a^{\frac{1}{2}} b^{\frac{1}{4}} a^{\frac{1}{8}} b^{\frac{1}{16}} a^{\frac{1}{32}} b^{\frac{1}{64}} \&c. =$$

$$a^{\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\&c.} \times b^{\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\&c.}$$

$=$ (becaufe the converging feries $\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\&c. = \frac{2}{3}$ and the converging feries $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\&c. = \frac{1}{3}$) $a^{\frac{2}{3}} b^{\frac{1}{3}} = \sqrt[3]{a^2 b}$ the fame as above, and it is here evident that the feries converge.

III. If $x^2 - 2ax = b$, we have $x^4 - 2ax^2 = bx$, and therefore

$$x = \sqrt{a+\sqrt{bx+a^2}} = \sqrt{a+\sqrt{a^2+b\sqrt{a+\sqrt{bx+a^2}}}}$$

$$= \sqrt{a+\sqrt{a^2+b\sqrt{a+\sqrt{a^2+b\sqrt{a+\sqrt{bx+a^2}}}}}} =$$

$$= \sqrt{\ldots + \sqrt{\ldots}}$$ [illegible]

IV. Let $x^3 + ax = b$, therefore $x = \frac{1}{a}b + \ldots$ [illegible]

[illegible]

and put $a = \ldots$ [illegible] becomes [illegible]

The computation in these series will be plain in the following Example. Let $x^3 - \ldots x = - 3$, to find x. Compare with the last theorem, and we have [illegible]

[illegible]

[illegible], which differs from the value of x but by 1 in the last place, and had we proceeded we might have had any degree of accuracy. Though what I have done here may not be of great importance, it may still be a[illegible] as it confirms Bernoulli's solutions.

ARTICLE IV.

Answers to the Mathematical Questions proposed in

ARTICLE XLIV. Vol. II.

I. QUESTION 212, *answered by Mr.* Gregory, *of Cav*[illegible]

WHEN it is required to determine the sun's altitud[e at] *six*, and his amplitude at rising, from the latitu[de and] *declination* given, the two following analogies are generall[y made] *use of*:

1. *As radius* : *sine latitude* : : sine of sun's dec. : sine alt. at

2. *As cos. latitude* : radius : : sine of sun's dec. : sine ampl

From the nature of which analogies it is manifest, that b[oth the] *altitude at 6*, and the amplitude will increase, as the latit[ude] *the same* rate as the declination increases. And as neit[her]

altitude nor amplitude increases very fast or very irregularly, the former varying from 0° in lat. 0°, to about 15° in lat. 90°; and the latter from 15° 5′ 24″ (= the declination) in lat. 0°, to about 30° in lat. 60°, the method of Trial and Errour, may here be adopted with sufficient accuracy and success.

Now, it is obvious upon slight consideration, that the latitude sought must be between 45° and 55°: suppose therefore, that the latitude is 50°; then, applying the above analogies, we get 11°. 30′. 13″ for the altitude at 6, and 23°. 53′. 30″ for the amplitude: the sum of these and the declination is 50°. 29′. 7′ instead of 50°; the errour, therefore, is 29′. 7″ or 1747″ in excess. Again, suppose 52° the latitude; then, we obtain, 11°. 50′. 17″ for the altitude, and 25°. 0′. 54″ for the amplitude: the sum of these and the declination is 51°. 56′. 35″, instead of 52°; hence the errour arising from this assumption is 3′. 25″ or 205″ in defect. Whence by the common rule we find the north latitude = 51·78996 or 51°. 47′. 24″ nearly. The altitude at 6, corresponding to this latitude, and declination 15°. 5. 24″ is 11°. 48′. 13″½, and the amplitude 24°. 53′. 25″½; the sum of these and the declination differ not more than 1′ from the latitude; thus confirming the truth of the result. In this solution no notice has been taken of the change in the sun's declination, between the sun's rising and noon of the given day.

The same, answered by the Proposer, Mr. Croudace.

Put the sine and cosine of the declination = a and b, and the sine of the latitude = x. Then, by Spherics, the sine of the altitude at 6, is = ax, and the sine of the amplitude = $a \div \sqrt{(1-x^2)}$. But, by *Emerson's Trigonometry, page* 21, sine alt. × cos. ampl. + cos. alt. × sine ampl. is = to sine lat. × cos. decl. — cos. lat. × sine decl. that is,

$$ax \times \sqrt{\frac{c-x^2}{1-x^2}} + \sqrt{(1-a^2x^2)} \times \sqrt{\frac{a^2}{1-x^2}} = bx - a\sqrt{(1-x^2)}, \text{ by}$$

putting $c = 1 - a^2$. This equation reduced gives,

$$(bx \div a) \times \sqrt{(1-x^2)} - \sqrt{(cx^2 - x)} - \sqrt{(1-a^2x^2)} + x^2 = 1.$$

Hence $x = {\cdot}784429$ answering to 51°. 40′. 5″ north latitude.

Answers were also received from Messrs. Barron, Baffham, Cunliffe, Davis, Lowry, May, Merones Minor, *and* Thornoby.

II. QUESTION 212, *answered by Miss* Susan May.

It is evident that the objects A, B, C, D are in the circumference of a small circle of the sphere, whose diameter may be

determined from the diſtances given in the queſtion. Now theſe diſtances are portions of a great circle of the ſphere, and conſequently the chords ſubtended by theſe diſtances may be found by reverſing ſome of the rules in *Dr. Hutton's Menſuration*, and the diameter of the circle may be found by *Prob.* 98, *of Emerſon's Algebra.*

But in the preſent caſe where the diſtances are ſo ſmall, that the difference between the arcs and chords are inconſiderable, we ſhall avoid the trouble of calculating the lengths of the chords, by uſing the arcs in their ſtead.——Now, by the Prob. above-mentioned, we obtain 33·1784 for the diameter of the circle, and if O (fig. 497, pl. 27.) be the centre of the earth, GF the height of the hill, FE a tangent to the earth at E and EH be $\perp$ to OF, we have, (by the ſim. $\triangle$s OFE, OEH,) $OE^2 = GH \times HF$: but OE is $= 3978{\cdot}87$, and $EH = 16{\cdot}5892 = \frac{1}{2}$ the diameter above found, therefore $OF = OE^2 \div OH = OE^2 \div \surd(OE^2 - EH^2) = 3978{\cdot}904$; conſequently $GF = {\cdot}034$ miles $= {\cdot}034 \times 5280 = 179{\cdot}5$ feet nearly the height required.

The ſame, anſwered by Merones Minor.

Here are given the four ſides of a trapezium inſcribed in a circle to find the diameter, which, *by Simpſon's Select Exerciſes, Prob.* 35, is $= 33{\cdot}1783$ miles. Then $25000 : 33{\cdot}1783 \div 2 :: 360° : 14'.\ 20''$, the $\angle$ at the centre of the earth ſubtended by the radius of the above circle; the natural ſecant thereof *minus* radius $= {\cdot}0000087$ miles for the height of the hill to radius 1, which, $\times (25000 \times 5280) \div (2 \times 3{\cdot}14159$ &c.) gives 182·8 feet, the height required, by Gardner's Logarithms. Or by uſing Hutton's Logarithms the height $= {\cdot}0000085$ miles (to radius 1) $= 178{\cdot}6$ feet.

This queſtion was alſo anſwered by Meſſrs. Barron, Buff ham, Cunliffe, Davis, Gregory, Lowry, *and* Thornoby.

III. QUESTION 213, *anſwered by Miſs* Suſan May.

Fig. 498, Pl. 27. Let ACE, ADF be the ſegments, draw the diameter AQI and join AC. Then, *by Rule V. of Dr. Hutton's Menſuration*, $(2CB + \frac{4}{3}CA) \times \frac{4}{15}AB =$ the area of the ſegment ACE, and ${\cdot}7854 \times AB^2$ is $=$ the area of the greateſt inſcribed circle; therefore by the Queſtion

$(2CB + \frac{4}{3}CA) \times \frac{4}{15}AB : {\cdot}7854 \times AB^2 :: 5 : 2$, or

$CB + \frac{2}{3}CA : AB :: 1 : {\cdot}407403$, or

$CB + \frac{2}{3}CA : \surd(CA^2 - CB^2) :: 1 : {\cdot}407403$. But (*Rule VI. ibid.*)

4CA − CB $= \frac{3}{2} \times$ Arc CAF $= 75$, wherefore

$+ \frac{1}{4}CB$, and $CA^2 = (\frac{75}{4})^2 +$ [illegible] $\times CB + \frac{1}{16} \times CB^2$. Theſe

val es

values being substituted in the above proportion it becomes
$\frac{15}{8}+\frac{7}{8}\times CB:\sqrt{\left(\frac{75^2}{16}+\frac{150}{4}\times CB-\frac{15}{16}\times CB^2\right)}::1:\cdot407403.$
Hence by means of a quadratic equation we obtain $CB = 18\cdot8$ nearly, $CA = 23\cdot45$, and $BA = 14\cdot025$; consequently $AC^2 \div AB = AQ = 39\cdot2$, the diameter of the least circle.

Again, *by Rule V. ibid.*
$2AI \times \sqrt{\left\{3AB \div (3AI - AB)\right\}}$ = the arc $DAF = 60$, or $4AI^2 \times 3AB \div (3AI - AB) = 60^2$. Therefore $AI^2 = (\frac{60}{2})^2 \times AI - 60 \times 5$; consequently $AI = 59\cdot075$ the diameter of the greater circle. Wherefore $(AI^2 - AQ^2) \times \cdot7854 = 1534$ square yards, the area included between the arcs of the circles, nearly.

The same, answered by Mr. Geo. Buffham, *Boston.*

Put $50 = l$, $\cdot7854 = p$, $AC = y$, and $CB = x$. Then by the common rule for finding the length of an arc, $8y - 2x = 3l$, or $y = \frac{1}{4}x + \frac{3}{8}l$; whence $AB = \sqrt{(y^2 - x^2)} = \sqrt{(\frac{3}{16}lx - \frac{15}{16}x^2 + \frac{9}{64}l^2)}$, and the area of the circle $AGBK = p \times (\frac{3}{16}lx - \frac{15}{16}x^2 + \frac{9}{64}l^2)$. By Dr. Hutton's Mensuration, 4to edit. Rule IV. for finding the area of a circular segment, the area $CAE = (\frac{14}{15}x + \frac{1}{5}l) \times \sqrt{\frac{3}{16}lx - \frac{15}{16}x^2 + \frac{9}{64}l^2}$. By the Question
$5:2::(\frac{14}{15}x+\frac{1}{5}l)\times\sqrt{(\frac{3}{16}lx-\frac{15}{16}x^2+\frac{9}{64}l^2)}:p\times(\frac{3}{16}lx-\frac{15}{16}x^2+\frac{9}{64}l^2)$; or
$5p\times(\frac{3}{16}lx-\frac{15}{16}x^2+\frac{9}{64}l^2)=(\frac{28}{15}x+\frac{2}{5}l)\times\sqrt{(\frac{3}{16}lx-\frac{15}{16}x^2+\frac{9}{64}l^2)}$.
This by reduction gives $x^2 - 3\cdot89636x = 279\cdot8783$, or $x = 18\cdot79$. Hence the diameter of the less circle is $= 39\cdot2$. And from the same rule the diameter of the greater circle is $= 59\cdot0707$, and the difference required $= 1533\frac{2}{3}$ square yards nearly.

Other solutions were received from Messrs. Barron, Davis, Gregory, Hill, Lowry, *and* Swale.

IV. QUESTION 214, *answered by Mr.* John Barron, *Spilsby.*

Fig, 499, Pl. 27. Put $D = AB = 6$ inches the diameter of the base, b = the altitude $= 4$ inches, m = a third part of the solidity $= 12\cdot5664$, and $FB = x$. Then, $D:b::x:bx \div D = DI$ the altitude of the part cut off. And by *Rule VI, Dr. Hutton's Mensuration*, the area of the part HBG is $= \frac{4}{3}x \times \sqrt{(Dx - \frac{5}{8}x^2)}$; therefore *by Rule to Prob.* 28, *page* 232, *ibid*, we have
$[\frac{4}{3}D \times \sqrt{(Dx - \frac{5}{8}x^2)} - \frac{4}{3}(D - x) \times \sqrt{(Dx - x^2)}] \times (bx \times D) = m.$

Hence

Hence by the method of trial and errour, x is found $= 3\cdot21$ nearly for the first man's share. Again, by taking $m = \frac{2}{3}$ of the solidity $= 25\cdot1327$, we find $x = 4\cdot4$ nearly for the second man's share.

Messrs. Buffham, Davis, Gregory, Hill, Lowry, *and* May, *also sent solutions to this question.*

V. QUESTION 215, *answered by Miss* Susan May.

In Fig. 500, Pl. 27. Let C be the centre of the earth's shadow, O the centre of the moon at the first observation, A the centre at the middle of the eclipse, or when the obscuration is the greatest, I that at the beginning, and H that at the end.—— About the centre C with the distance CO describe a circle DOBP for the boundary of the earth's shadow, and let CAP be drawn, which will be $\perp$ to HI, the relative path of the moon's centre. Join BC, BD, BO, PO, CO, CD, CH and CI. Then by the question, BD the distance between the illuminated cusps is given $= 30'.\ 15''$, therefore its half, BE, is given $= 15'.\ 7\cdot5'' = 907''\cdot5$, and the semidiameter BO is given $= 15'.\ 24'' = 924''$; therefore $OE = \sqrt{(BO^2 - BE^2)} = 173\cdot83''$, and, *by Lawson's 43rd Prop.* (vide Repos. No. X.) $OC = CP = BO^2 \div 2OE = 2455''\cdot65$.
Again as, $6^\circ : 15'.\ 24'' :: 1^\circ.\ 27'.\ 56'' : 224''\cdot84 = AP$; therefore $AC = CP - AP = 2230''\cdot81$ and $AO = \sqrt{(OC^2 - AC^2)} = 1026''\cdot52$. Moreover $CI = CO + OB = 3379''\cdot65$,
and therefore $AI = \sqrt{(CI^2 - CA^2)} = 2535''\cdot89$. Now as
$29'.\ 45''$ (the horary motion) : $60' :: \begin{cases} AI : 1h\ 25m.\ 14s. \\ AO : 0h.\ 34m.\ 30s. \end{cases}$ Hence
11h. 49m. 48s. + 34m. 30s. = 12h. 24m. 18s, the middle,
12h. 24m. 18s. — 1h. 25m. 14s = 10h. 59m. 4s, the beginning, and
12h. 24m. 18s. + 1h. 25m. 14s = 13h. 49m. 32s, the ending.

The same, answered by Merones Minor.

The figure being drawn as before, bisect OD by the $\perp$ Ch. Then in the $\triangle$ DEO we have DO, DE to find the $\angle$ DOE (COh) $= 79^\circ.\ 9'.\ 35''$, and in the $\triangle$ COh we have all the $\angle$s and the side Oh to find $CO = 2455''.\ 64$. Then say as $6^\circ : 1^\circ.\ 27'.\ 36'' :: 15'.\ 24'' : 224''\cdot84$, and $2455''\cdot64 - 224''\cdot84 = 2230''\cdot8 = AC$; also $CO + DO = CI = 3379''\cdot64$. Now in the two right $\angle$d $\triangle$s CAO, CAI we have given CA, CO, CI to find $AO = 1026''\cdot5$, and $AI = 2235''\cdot9$, which, by the given horary motion of the moon from the sun will be passed over

in 34′. 30″. 25 and 1h. 25′. 14″·37 respectively. Hence by addition and subtraction,

the beginning was at 10h. 59m. 3·88s.
the middle was at 12h. 24m. 18·25s.
and ending was at 13h. 49m. 32·62s.

⁎ In these solutions the moon's relative orbit has been considered as a straight line, and her motion therein as equable; since it was not the wish of the proposer to render the solution of the question more intricate by considering them otherwise.

Ingenious solutions to this question were received from Messrs. Davis, Gregory, Hill *and* Lowry.

VI. QUESTION 216, *answered by Mr.* John Andrew, *of Cark.*

CONSTRUCTION. Fig. 501, Pl. 27. With the given diameter DO describe a circle, and ⊥ thereto inscribe AB = the given base; join OB, to which add, *by the* 18, *V. of Simpson's Geometry*, the line Bg, so that Og × gB may be = to the rectangle under DO and the given difference between the ⊥ and rad. of the inscribed circle. From O, to the circle, apply OC = Og; join AC, CB and ACB will be the △ required.

DEMONSTRATION. Join DC, CO, and drop the ⊥ CP; make OQ = OB, and draw QG ∥ AB. Now, it is well known, that Q is the centre and GP the radius of the circle inscribed in the △ ACB. Moreover by similar triangles
OD : OC = Og : : CQ = Bg : CG, hence
Og × gB = OD × CG, and, by the construction,
Og × gB = OD × the given difference;
therefore CG is equal to the given difference. *Q. E. D.*

The same answered by Mr. James Cunliffe, *of Bolton.*

ANALYSIS. Let ACB be the required △ inscribed in the given circle, DO a diameter of the circle bisecting the base AB in E; join OB and OC.

Upon OC take OQ = OB, and Q is the centre of the inscribed circle, as is very well known. Draw QG ∥ to AB meeting the ⊥ CP in G, and join DC. The △s ODC, and QCG are right angled and similar, therefore
OD: OC = OB + CQ : : CQ : CG; whence by Eu. 16. VI.
CQ × (CQ + OB) = OD × CG; now OD, OB and CG are all given, and CQ may be found *by Prob.* 18. *Book V. Simpson's Geometry*, and from thence OC becomes known.

The ſame anſwered by Mr. W. Peacock, *Birmingham.*

In a circle, whoſe diameter is the given line DO, inſcribe the chord AB, $\perp$ to DO and equal to the given baſe. With the centre O and diſtance OA, or OB, deſcribe the circle AIQB cutting DO at I. Make OD : OF :: IF : the given difference (Eu. VI. 29. Playfair's Ed.), and from O, to the circle apply OC = OF, and join AC, BC, ſo ſhall ACB be the $\triangle$ required.

For it is well known that the circle AIB is the *locus* of the centre of the inſcribed circle, and if CP be drawn $\perp$, and QG $\|$ to AB, we ſhall have by ſim. $\triangle$s, OD : OC = OF : QC = FI : CG; therefore it is manifeſt that CG is = to the given difference.

The ſame, alſo by Mr. James Whalley, *of Bolton.*

Analysis. Suppoſe DO the diameter of the given circle, AB, $\perp$ to DO, the given baſe, and ACB the $\triangle$ required. Draw C*d* $\|$ to AB, then will *d*E repreſent the $\perp$ in which take $dn =$ the given difference. Now (vide No. 3. of the Student) $En \times On = dn \times OE$, but *dn* and OE are both given, therefore (by *Simpſon's Geo.* 18. V.) produce OE to *n* ſo that $On \times nE$ may be = to the given rectangle $dn \times OE$, then will E*d* be given, and all the other parts of the diameter will from thence be given, and the method of Conſtruction evident enough.

Very neat anſwers to this queſtion were alſo received from Meſſrs. Burdon, *the propoſer,* Davis, Gregory, Hill, Lowry, *and* Swale.

VII. QUESTION 217, *anſwered by the propoſer, Mr.* Boſworth.

The beſt method perhaps of anſwering Queſtions of this nature is, firſt, to find the centre of gravity of the lever, and then, ſuppoſing the whole matter it contains to be collected in that point, to find the common centre of gravity of the lever and the globe attached to it, which will be the place of the fulcrum. To apply this method in the preſent inſtance: the lever being equally thick and denſe throughout, its centre of gravity is in the middle, that is, the length being 10 feet, at the diſtance of 5 feet from each end. A globe of lead, $4\frac{1}{2}$ inches diameter, weighs 17lbs, and as the weights of ſimilar ſolids are in proportion to the cubes of their like dimenſions we have $(4\frac{1}{2})^3 : 17 :: 6^3$, (the cube of the diam. of the given globe) : 47·8339lbs. weight of the given globe. Now, as the diſtance of the common centre of gravity of theſe bodies, and conſequently of the fulcrum, from each end, is inverſely as the weights, we have as 40 (weight of the lever) +

47·8339

47·8339 (weight of the globe) : 5 feet : : 40 : 2·27702515 feet, distance of the fulcrum from the end at which the globe hangs.

The same, answered by Miss Susan May.

Fig. 502, Pl. 27. Let ADHE be a section of the lever, and let G be the point required. Take FG = GH, and draw FB ∥ to HD. Then the equal parts GCDH, GCBF on the different sides of the fulcrum G, will keep each other in equilibrio; therefore the weight suspended at H must be a counterpoise for the remaining part of the lever, that is for the part ABEF.

Let m be the centre of gravity of the part ABEF, then, by mechanics, the force of the part ABEF, to turn the lever about the fulcrum G, will be the same as a weight equal thereto placed at m; consequently as the weight of the part ABEF is to the distance cd or GH, so is the weight of the lead to the distance mc.—Now since $am = mb$, and $bc = cd$, mc is = half the length of the lever 5 feet, and the weight of the part ABEF is $= 8mb$, therefore, if w be = to the weight of the lead (= 46·326 lbs.) we have $8mb : 5 - mb :: w : 5$, and, by reduction, $mb = 5w \div (40 + w) = 2{\cdot}68321$, consequently $bc = 5 - mb = 2{\cdot}31679$ the distance required.

The same, answered by Mr. John Barron, *of Spilsby.*

The weight of a leaden ball is $= \frac{2}{9}$ of the cube of its diameter, therefore $6^3 \times \frac{2}{9} = 48$ lbs. the weight of the given ball.

Let ad be = 10 the len. of the lever, $cd = x$, the distance from that end of the lever, from whence the ball is hung, to the fulcrum; then $10 - x = ac$ the other part of the lever. Since the weight of every foot of the lever is evidently = 4lb, $4x$ will be the weight of the part GD, and $40 - 4x$ the weight of the part AG. The distance of the centre of gravity of the part DG from the point c is $= \frac{x}{2}$, and that of the part AG from the same point c is $\frac{10 - x}{2}$. Hence, by the nature of the question $\frac{x}{2} \times 4x + 48x = \frac{10 - x}{2} \times (40 - 4x)$. From this equation $x = \frac{200}{98} = 2{\cdot}2727$ feet.

The same, answered by Mr. George Buffham.

Let ad represent the lever, w the globe of lead, the weight of which is found to be 46·326 lbs, and let c be the point where the

the fulcrum is to be placed. Put $cd = y$, then $ac = 10 - y$, the weights of which will be $4y$ and $40 - 4y$. By the property of the lever $(40 - 4y) \times (5 - \frac{1}{2}y) = 4y \times \frac{1}{2}y + 46{\cdot}326y$; whence $y = 2{\cdot}3168$ the diftance from the end d to the point required.

The fame, anfwered by Mr. David Davis.

Let F (fig. 503, pl. 27.) be the fulcrum on which the lever AB and weight W are fuppofed to reft *in equilibrio*: then, fince the lever is equally thick throughout, if FC be equal to FA, the *momenta* of thefe two portions of the lever will be equal the one to the other, and we have to determine when the momentum of the part CB is equal to that of the weight W, both confidered with refpect to the fulcrum F. To perform this generally, put x for FB, y for AF or FC, s for 4 lbs, the weight of 1 foot in length of the lever, and w for the weight fufpended from A: then yw will denote the momentum of W; and CB being equal to $x - y$, the diftance FG of its centre of gravity from F being $y + \frac{x-y}{2} = \frac{x+y}{2}$; we have $(x - y) \times s \times \frac{x+y}{2} = \frac{sx^2 - sy^2}{2}$, for the momentum of CB. Hence this equation, $yw = \frac{sx^2 - sy^2}{2}$; which reduced gives $x = \sqrt{(y^2 + \frac{2wy}{s})}$. And from this general equation, any three of the four quantities x, y, w, or s, being given, the fourth may readily be found.

In the cafe before us, where $AB = 10$, we have $x = 10 - y$; which value of x equated with the foregoing, gives $10 - y = \sqrt{(y^2 + \frac{2wy}{s})}$; whence by proper reduction we foon obtain $y = \frac{100s}{20s + 2w} = \frac{200}{40 + w}$, s being equal to 4......Now, it is known, by repeated experiments, that a ball of lead of $4\frac{1}{4}$ inches diameter weighs 17 lbs; wherefore, we have, by the nature of fimilar folids, $(4\frac{1}{4})^3 : 6^3 :: 17$ lbs : $47{\cdot}84391$ lbs, the weight of the given globe, or the value of w; which being fubftituted for it in the final equation above, gives $y = 2{\cdot}277024898$ feet, $=$ AF, the diftance of the fulcrum from the end A.

This queftion was likewife very ingenioufly anfwered by Meffrs. [illegible]ffe, Gregory, Hill, Lowry, Marrat, Merones Minor, Pea- [illegible] *and* Swale.

VIII.

VIII. QUESTION 218, *anſwered by Mr.* Cunliffe.

CONSTRUCTION. Fig. 504, Pl. 27. Make a right angled △ BGO′ ſuch, that the leg BG ſhall be to the hypothenuſe BO′ in the given ratio of the baſe to the diameter of the circumſcribing circle: at B take BL ⊥ to BG and ſuch that BG may be to BL in the given ratio of half the difference of the ſegments of the baſe to the perpendicular, and draw GL. Then with the centre O′ and radius O′B deſcribe a circle cutting BG produced in A and the line GL in C; join AC, BC, and the △ ACB ſhall be ſimilar to that required.

DEMONSTRATION. Upon AB drop the ⊥ CP; then becauſe of the parallels PC, BL; GB : BL :: GP : PC, the given ratio of half the difference of the ſegments of the baſe to the perpendicular, by the conſtruction; and the reſt is evident. Therefore the △ may be conſtructed either with the radius of the inſcribed circle, or any other datum.

The ſame, anſwered by Merones Minor.

ANALYSIS. Suppoſe ACB the required △, DGI, ⊥ to AB, the diameter of the circumſcribing circle, and O the centre of the inſcribed circle. Draw CP ⊥ to AB and join CG. Then, the ratio of AB to DI being given, the vertical ∠ ACB, is given; and OM, the radius of the inſcribed circle, being given, CO is given. Hence this

CONSTRUCTION. Upon any aſſumed line *di* deſcribe a circle and inſcribe *ab* ⊥ thereto, ſo that *ab* : *di* = AB : DI; take *ep* : *eg* = GP : CP, of any magnitude at pleaſure; compleat the parallelogram *epcg*, and draw the diagonal *ec*C to cut the circle in C; join C*a*, C*b*, C*i*; in C*i* take CO of the given length, as found in the analyſis, and deſcribe the circle HMN, with the given radius of the inſcribed circle; finally draw AB ∥ to *ab*, to touch the circle HMN, and cutting C*a*, C*b*, in A and B; ſo ſhall ACB be the △ required.

The ſame, anſwered by Mr. W. Peacock.

On any line DI as diameter deſcribe a circle and apply AB ⊥ to the diameter and of ſuch a length that its ratio to DI may be the ſame as the given ratio of the baſe to the diameter. Let G be their point of interſection, draw DE ⊥ DG, and take 2DE to DG in the other given ratio. Draw GE to meet the circle at C, join AC, BC; the △ ACB is evidently ſimilar to the required one.

The

The same, answered by Mr. James Whalley.

Analysis. Suppose the required △ ACB to be circumscribed by the circle ADBI, whose diameter DI is ⊥ to AB the base of the △. Let O be the centre of the inscribed circle; upon AB demit the ⊥s CP, OH, and draw CG, G being the middle of the base, draw also OA, OB. Now the vertical ∠ ACB is given, because the ratio of AB : DI is given; and, the ratio of CP : PG being given, the △ CPG is given in species, hence the ∠ CGB is given; therefore the △ ABC is given in species; wherefore the △ OAB is given in species; but OH is given in magnitude, therefore the △ OAB is given in magnitude, consequently AB is given, and a construction is evident.

Neat solutions were given to this question by Messrs. Barr, Davis, Gregory, Hill, Lowry, May, *and* Swale.

IX. QUESTION 219, *answered by Miss* Susan May.

Fig. 505, Pl. 27. Let EE′ be the equatorial diameter, NS the polar diameter, C the centre of the earth, and P the place where the apparent latitude is taken. Join PC and draw PB ⊥ to the horizon at P; then (if I rightly comprehend the proposer's meaning) there is given the ∠ EBP, the measure of the apparent latitude, to find a simple and accurate expression for the ∠ ECP, the latitude at the centre.

On EC demit the ⊥ PA, and draw BF ∥ to PC. Then by Prop. XV. *Hutton on the ellipse*, $E'E^2 : NS^2 :: AC : AB :: AP : AF ::$ tang. ∠ PBA : tang. ∠ FBA, or its equal the ∠ PCA. Wherefore as the square of the equatorial diameter is to the square of the polar diameter, so is the tangent of the apparent latitude to the tangent of the latitude at the centre.

The same, answered by Mr. T. S. Evans, *the proposer.*

Let EPN be a quarter of the meridian: EC the equatorial radius which put $= a$; NC the semi-axis $= b$. Let PB $= n$ be drawn ⊥ to PT a tangent to P, the place whose apparent latitude is given, the sine of which put $= s$; CA $= x$, and AP $= y$; then by the property of the ellipsis $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$, likewise

$\frac{a^2 - x^2}{x} = AT$; and $AF^2 : AP^2 :: AP^2 : AB^2$ or $\left(\frac{a^2 - x^2}{x}\right)^2$: $\frac{b^4}{a^4}(a^2 - x^2) :: \frac{b^2}{a^2}(a^2 - x^2) : \frac{b^4 x^2}{a^4} = AB^2$; now $x^2 = a^2 - \frac{a^2}{b^2} y^2$, therefore ſubſtituting this for x we get $AB^2 = \frac{b^4}{a^2} - \frac{b^2}{a^2} y^2$: but $AB^2 + AP^2 = BP^2 = \frac{b^4}{a^2} - \frac{b^2}{a^2} y^2 + y^2 = n^2$, and $y = sn$, therefore $n^2 = \frac{b^4}{a^2} - \frac{b^2}{a^2} s^2 n^2 + s^2 n^2$, whence

$n = \sqrt{\frac{b^4}{a^2 + b^2 s^2 - a^2 s^2}}$, conſeq. $y = \sqrt{\frac{b^4 s^2}{a^2 + b^2 s^2 - a^2 s^2}}$ and $x = \sqrt{\frac{a^4 - a^4 s^2}{a^2 + b^2 s^2 - a^2 s^2}}$.

Now in the $\triangle$ APC, if AC be made radius, AP will be the tangent of PCA $= \frac{y}{x} = \sqrt{\frac{b^4 s^2}{a^4 - a^4 s^2}} = \frac{b^2}{a^2} \times \frac{\text{ſine lat.}}{\text{coſine lat.}}$; whence the tangent of the angle PCA, or of the latitude reduced to the earth's centre $= \frac{b^2}{a^2} \times$ tangent of the apparent latitude; which is a very ſimple expreſſion. The reader may now conſult *Bezout's* Navigation at the end.

This ſolution being general, for any proportions of the earth's diameters, it may not be amiſs to notice the preſent ſtate of equations for reducing the apparent latitude to that at the centre.—By comparing the meaſurement of *Maupertuis* in lat. 66°. 20′N with that of *Maſon* and *Dixon* in lat. 39°. 12′N, we ſhall have the proportion of the earth's diameters $\frac{140}{141}$; that of *Maupertuis* with *Bouguer* at the equator, we ſhall obtain $\frac{216}{217}$; and laſtly that of *Caille* in lat. 45°N with his in lat. 49°. 23′N we ſhall have the proportion $\frac{321}{322}$; now

$\left.\begin{array}{l}\frac{140}{141}\\ \frac{216}{217}\\ \frac{321}{322}\end{array}\right\}$ gives the maximum of reduction of the apparent latitude to that at the centre. $\left\{\begin{array}{l}24'.\ 28''\\ 15.\ 23\\ 10.\ 42.\end{array}\right.$

Hence

Hence it appears how uncertain we are reſpecting the preciſe quantity of this equation, and there are other proportions of the earth's diameters which run to a far greater extreme than theſe.—From this alſo appears the great utility of the preſent Trigonometrical Survey carrying on both in England and France.

As I do not find that *Mayer* has mentioned the proportion he uſed in calculating his Table pa. LXXV; it appears, by an inverſe method of calculation, that he uſed the proportion $\frac{230}{231}$; and *La Lande* uſed the proportion $\frac{229}{230}$; which is the ſame as *Sir Iſaac Newton* found from the meaſurements of *Norwood*, *Picart* and the two *Caſſinis*; ſee the Principia B. 3, P. 19.

Meſſrs Gregory, Hill, Lowry, *and* Thornoby *ſent neat ſolutions to this queſtion.*

X. QUESTION 220, *anſwered by Mr.* John Lowry.

Fig. 506, Pl. 27. Through I draw IL parallel to AC, meeting AB at L, and join LH. Draw HI to meet AC at G, and CI to meet AB at M, and join HM. On HL take HO = GC, and draw OQ parallel to IH meeting MH at Q; draw alſo QV parallel to OH meeting CB at V. Then it is manifeſt, *from Prop. IV of Profeſſor Playfair's Paper on Poriſms*, that if any two lines whatever, as HE, IE be inflected from the points H, I, to the ſtraight line AB, and meet QV in S and AC in F, QS will be equal to FC; and becauſe the angle FCD is given, the triangle FCD will have a given ratio to the rectangle FC·CD, or its equal QS·CD; therefore the rectangle QS·CD is of a given magnitude. The problem is therefore reduced to the following, viz. Through the given point H, to draw the ſtraight line DHS ſo, that the ſum or rectangle of QS, CD may be a given magnitude.

The firſt caſe has been done by *Simpſon in his Geometry and Select Exerciſes*, and by *Burrow in his Diary for* 1776, and it may be done rather differently as follows. Draw HX parallel to CV, and HY parallel to QV; then it is evident that the ſum of SX and DY is given; and by ſimilar triangles DY : HY : : HX : SX, or DY·SX = HY·HX = a given rectangle; therefore the ſum and rectangle of DY, SX are given to determine them, the method of doing which is well known.

The rectangle remaining conſtant, the ſum of DY and SX will evidently be the leaſt poſſible when DY is = SX, in which caſe the ſum of QS and CD, or CF and CD will be a maximum.

The ſecond caſe is the ſame as the celebrated problem of the Ancient Geometricians, *De Sectione Spatii*, and may be done as follows. CD = CY — DY and QS = QX — SX; therefore CD·QS = CY·QX — QX·DY — CY·SX + DY·SX: but

DY·SX

DY·SX is given, as is shewn above, it is therefore evident QX·DY + CY·SX is given; let this be taken = R × and take HW to HY as CY to QX, and draw WU parall DY to meet HD (produced if necessary) at U. Then, by sin triangles, it is evident that QX·WU is = to CY·SX, there DY + WU is = to R, and DY·WU = HX·HW. Where the sum and rectangle are given to determine each, the same before.

DY + WU will be the least when SX is = to WU which case the rectangle CD·CF will be a maximum.

If the ratio of CD to CF had been given, the problem would been reduced to the *Sectio Rationis* of the Ancients. Or, bec CY and QV are given lines, DY·SX a given rectangle, and QS a given ratio, CD and QS may be determined by *Prop. Case 2nd. Geometrical Sections in the Repository*, and this per suggests as simple a solution to the celebrated problem of ancients as any that has yet been given.

The same, answered by Mr. I. H. Swale.

I. When the sum of CF, CD is given. Fig. 507, Pl. 2

CONSTRUCTION. In the sides CA of the given △ A take CG = the given sum, join CH, GI, which produce to in P; draw HO, PO, parallel to CB, CA meeting in O, ON : OH in the ratio of the rectangle HC × IP to IG × and join HN; produce HO to AB at T, also draw PR para to HO meeting AB at R; produce OP to meet AB at S, and d IV parallel to OP meeting AB in V, and join IR.

Then, from the point T, *by Prob.* 37, *Simp. Geo.* draw right line TWQU, meeting the given lines IR, HN, IV, W, Q, U and making WQ : QU = RS : SV. Draw meeting AB, AC, in E, F; draw EH meeting BC in D, DF, and the △ EDF will be inscribed as required.

DEMONSTRATION. Let DE, EF, meet RP, SP, in and L; join KL, RL, and produce RL to meet IV at M.

By Construction,

WQ : QU = RS : SV = RL : LM, per the parallels SL, V therefore the lines TQ, RL, are parallel; then,

TR : RE = QL : LE = HK : KE, per the parallels RK, TH and therefore the lines KL, HQ, are also parallel; wherefore

KP : PL = HO : ON, = IG × HP : HC × IP, by the co

Now, because PK is ∥ to CD, and PL ∥ to FG, it follows HP : HC = PK : CD and IP : IG = PL : FG; therefore

FG × PK : CD × PL = IG × HP : HC × IP = KP.PL, from abov

 He

Hence $TG : CD = PK \times PL : PK \times PL$, or $FG = CD$.
Therefore $CF + CD = CF + FG = CG =$ the given sum. *Q. E. D.*

II. When the $\triangle$ FCD is of a given magnitude. Fig. 508, Pl. 27.

CONSTRUCTION. Demit upon the side AC, of the given $\triangle$ ACB, the perpendicular BR, and make CV^2 : given magnitude $= BC : BR$. Join CH, CI, meeting AB at K and G; draw HS, IT, parallel to CB, CA and meeting AB at S and T; and make $CU^2 : CV^2 = HK \times IG : CH \times CI$. Then, by Wales' Determinate Section, cut ST at E so that $EK \times EG : ES \times ET = CU^2 : SH \times TI$. Join EH, EI, meeting CB, CA, at D and F; join DF, so shall the $\triangle$ CDF be inscribed as was required.

DEMONSTRATION. Draw KL, GM ∥ to BC, AC, meeting DE, FE in L and M.—By reason of the parallels,
$EK : ES = KL : SH$ and $EG : ET = GM : TI$, therefore
$KL \times GM : SH \times TI = EK \times EG : ES \times ET = CU^2 : SH \times TI$, by constr.
therefore $KL \times GM = CU^2$.—Now,
$CH : HK = CD : KL$ and $CI : IG = CF : GM$, therefore
$CD \times CF : KL \times GM = CH \times CI : HK \times IG = CV^2 : CU^2 = KL \times GM$;
therefore $CD \times CF = CV^2$.—But, by known properties,
$CD \times CF$ (CV^2) : $\triangle$ DCF $=$ CB:BR $= CV^2$: given magnitude.
Therefore the $\triangle$ DCF is $=$ the given magnitude. *Q. E. D.*

Messrs. Cunliffe *and* Hill *also sent neat solutions to this question.*

XI. QUESTION 221, *answered by Miss.* Susan May.

Fig. 509, Pl. 27. Let E be the place of the earth, M that of the moon, S that of the sun, and A the point where their attractions are mutually equal. Then, *by Art.* 382, *Simpson's Fluxions*, the attraction of spherical bodies are directly as the quantity of matter, and inversely as the square of the distance. Consequently the ratio of the distances of the point A from the points E, M, S, will be given. Let EA, MA and SA be drawn; then because E and M are given points, and EA to MA a given ratio, the *locus* of the point A is a circle given in magnitude and position, and whose centre is in the line EM produced. And, because S is also another given point and MA to SA a given ratio, the *locus* of the point A is also another circle given in magnitude and position, and whose centre is in the line SM produced. Consequently, the intersection of these circles will determine the point A. The method of describing these circles may be seen at page 331 of *Simpson's Algebra*, and thence the calculation will also be evident.

If

If the circle HAF revolve about its diameter HF, and thereby generate a ſpherical ſuperficies, it is evident that at any point whatever, in that ſuperficies, the attraction of the earth and moon will be equal; and in like manner, if the circle LAR generate a ſpherical ſuperficies by revolving round its diameter LR, any point in the ſuperficies will be equally attracted by the ſun and moon. Now, if we conceive theſe two ſuperficies to interſect each other, the figure of the ſection will be a circle whoſe diameter is AB, the line joining the interſections of the circles HAF, LAR. Conſequently any point whatever in the circumference of that circle is equally attracted by the three bodies.

The ſame, anſwered by Merones Minor.

Let A, B, C repreſent the given magnitudes of the ſun, earth, and moon, reſpectively, and a, b, c, their reſpective diſtances from the point required. Then, the centripetal force being inverſely as the ſquare of the diſtance, which is the GRAND LAW OF NATURE. And, by *Emerſon on Central Forces, Prop. I.* the centripetal force is as the quantity of matter in the attracting body; that is as the given magnitude in the preſent caſe. Therefore $A \div a^2 = B \div b^2 = C \div c^2$, or $\sqrt{A} \div a = \sqrt{B} \div b = \sqrt{C} \div c$, ſo that the queſtion becomes, "To find a fourth point P, from whence, lines drawn to the three given points may obtain a given ratio;" which is *Simpſon's* 53rd. *Prob. App. Alg.; or the* 56th *Prob. in his Select Exerciſes.*

XII. QUESTION 222, *anſwered by* Merones Minor.

Put $A = 10$ acres, $= 435600$ feet, $d = 4$, $b = 3$, $g = 16\frac{1}{12}$, $x =$ any variable altitude of the ſurface above the bottom of the apertures, and $t =$ time ſought. Then, by *Dr. Hutton's Select Exerciſes, Prob.* 26, $t = \dfrac{3A}{4b\sqrt{g}} \times \dfrac{\sqrt{d} - \sqrt{x}}{\sqrt{dx}}$. But when $x = 0$ this expreſſion is infinite, as there obſerved; if we ſuppoſe $x = \frac{1}{100}$, then $t = 252083''$, very near, $=$ 70h. 1m. 23s. the time of exhauſting the pool to within ·01 foot of the bottom of the apertures; ſuppoſing the velocity $=$ *that* due to the whole altitude.

The ſame, anſwered by Miſs Suſan May.

This queſtion ſeems deſigned as an example to *Prob. XXVI. of Dr. Hutton's Practical Exerciſes concerning forces,* where it is

 ſhewn

shewn that if x represent the variable height of the water at any time, b the breadth of the cut (3 feet), d the whole depth (4 feet), A the area of the surface of the water (= 435600 feet), and $g = 16\frac{1}{12}$ feet, the general time of sinking the surface to the depth x will be $\frac{8A}{2b\sqrt{g}} \times \frac{\sqrt{d} - \sqrt{x}}{\sqrt{dx}}$. But when x is o this expression is infinite, which shews that the water can never be exhausted so as to be on a level with the bottom of the aperture.

By taking $x = \frac{1}{12}$, we obtain, by substituting, the particular values in the above expression, the time of exhausting to $\frac{1}{12}$ of an inch of the bottom = 624551″ nearly. Now this is the time of exhausting through one of the apertures: but since an equal quantity is discharged through each, the time will be only one half of that above, that is = 312275″·5 = 86h. 34m. 35½s.

Mr. GREGORY *of Cambridge* also refers to the same problem in *Dr. Hutton's Select Exercises*, and adopts the general expression $t = \frac{8A}{4b\sqrt{g}} \times \frac{\sqrt{d} - \sqrt{x}}{\sqrt{dx}}$, for the time of sinking the surface to the depth x in both apertures, whence when $x = \frac{1}{100}$ he deduces the value of t = 252082·88″. He then observes that when one value of t is thus found, others may be readily ascertained, since they will vary as $\frac{\sqrt{d} - \sqrt{x}}{\sqrt{x}}$. Thus when $x = \frac{1}{10000}$, $\frac{\sqrt{d} - \sqrt{x}}{\sqrt{x}} = 93{\cdot}868328$; but when $x = \frac{1}{100}$, $\frac{\sqrt{d} - \sqrt{x}}{\sqrt{x}} = 19{\cdot}$: therefore, as 19· : 93·868328 :: 252082·88″ : 1245400·97174″ = 345h. 56m. 40·97s, time of exhausting to the depth of $\frac{1}{1000}$ of a foot.

The same, answered by Mr. John Barron, *Spilsby.*

Put d = the depth of the apertures, b = the breadth of one of them, $g = 16\frac{1}{12}$, a = area of the pool, and x = any variable altitude of the water above the bottoms of the apertures. Then by the Principles of Hydrostatics $2\sqrt{(gx)}$ = the velocity of the effluent

effluent water, and $2b\dot{x}\sqrt{(gx)}$ will, therefore, express the fluxion of the quantity of water issuing through one of the apertures in a second of time. The fluent of which is $\frac{4bx}{3}\sqrt{(gx)}$, the quantity of water issuing through one of the apertures in a second; wherefore, $-a\dot{x} \div \frac{4bx}{3}\sqrt{(gx)} = \frac{3a}{4b\sqrt{g}} \times \frac{-\dot{x}}{x^{\frac{3}{2}}}$ = the fluxion of the time, the fluent of which is $\frac{3a}{2b\sqrt{(gx)}}$. But when $x = d$, the time is nothing, therefore the correct fluent is

$$\frac{3a}{2b\sqrt{g}}\left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{d}}\right)$$ = the time through one of the apertures, and therefore $\frac{3a}{4b\sqrt{g}}\left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{d}}\right)$ = the time through both apertures. Now when $x = 0$, this expression for the time becomes infinite, which shews that the surface of the water in the pool can never arrive at the level of the bottoms of the apertures.

And thus the question was answered by Mr. Cunliffe. *It was also answered by Mess*rs. Bosworth, Davis, Hill, *and* Lowry.

To XIII. QUESTION 223, *we have received no answer.*

XIV. QUESTION 224, *answered by Mr.* James Cunliffe.

Let ABC (fig. 510, pl. 27.) be the $\triangle$, O the centre of its inscribed circle, through which and the angular points A and C draw the right lines AE, CD to terminate in the opposite sides BC, AB. These lines, as is well known, will bisect the respective angles CAB, and ACB. Let the parts of the weight sustained by the fulcrums at the respective angular points be denoted by A, B, and C. It appears from *Emerson's Mechanics, Prop.* 43, *Sect.* 5, that O is the common centre of gravity of the parts of the weight sustained by each of the fulcrums: moreover it appears from Prop. 45, of the same, that D is the centre of gravity of the parts denoted by A and B; and E that of the parts denoted by B and C. Wherefore, since CD bisects the $\angle$ ACB, by the nature of the centre of gravity and Euc. 3. VI.

AC : CB : : AD : DB : : B : A, whence
$A = (BC \div AC) \times B = \frac{4}{5}B$. And for the same reason
AC : AB : : B : C $= (AB \div AC) \times B = \frac{13}{15}B$.
Then by the question $A + B + C = \frac{4}{5}B + B + \frac{13}{15}B = \frac{8}{3}B = 100$.
Whence $B = 37\frac{1}{2}$lb, $A = \frac{4}{5}B = 30$lb, and $C = \frac{13}{15}B = 32\frac{1}{2}$lb.

When each of the props sustain an equal part of the weight, the point O, the place thereof, will be determined by the intersection of the lines CD, AE drawn to bisect AB, BC, in the points D and E respectively.

If O, the place of the weight, be any other given point within the △ ABC, the parts of the weight sustained by props at each of the angular points may be determined as follows, viz. by drawing through the given point the right lines CD, AE from the ∠s C and A to the opposite sides. For it is evident from the above quoted authority that D is the centre of gravity of the parts A and B, and E that of the parts B and C, the ratios of which hence become known.

The same, answered by Mr. Gregory, *the proposer.*

In the solution of this question it is necessary, in the first place, to enquire, when a plane triangle is supported by three fulcra at the the angular points, and a weight W is laid upon it at any other point, what is the ratio of the pressures on each fulcrum? To ascertain this, let ABC (fig. 511, pl. 27.) be the triangle, and D the point where the weight is placed; produce BD to F, and from the points B and D let fall the perpendiculars BH, DG, upon the side AC. Then, if we suppose the triangle to revolve about AC as an axis, the point F may be considered as a fulcrum to the power P at B, and weight W at D: therefore, P : W : : DF : BF : : DG : BH, by similar triangles. Consequently $P = \frac{DG \times W}{BH}$. But W and BH are constant, therefore $P \propto DG$, and of course $P \propto DG \times AC$, or $\propto \frac{DG + AC}{2}$; that is, *the pressure at B is as the area of the triangle DAC opposite.* In the same manner it might be shewn, that the pressures at A and C, are as the areas of the triangles BCD and BAD respectively.

Hence, the first requisite of the question is easily determined: for when the weight is laid at D the centre of the inscribed circle, the perpendiculars DF, DI, and DL, let fall on the sides, will be equal to each other, and the areas of the triangles ACD, ABD, BCD, will be as their bases AC, AB, BC; therefore, by what has been shewn above, the pressure upon the angles A, B, C, will BC, AC, and AB, or as 26, 30, and 24 respectively.

It

It is alfo manifeft from the foregoing, that, when the preffure is the fame on each fulcrum, the point D muft be fo chofen that lines drawn from it to the angular points fhall divide the triangle into three equal parts. In this cafe the point D may be found by the following

CONSTRUCTION: Bifect AB in the point E by the line CE, make AP = $\frac{1}{3}$AC, and draw PK parallel to AB: the point D in which CE and PK interfect each other, is the point required. ....For, fince CP = $\frac{2}{3}$AC, and the triangles CPD, CAE are fimilar; fimilar triangles being as the fquares of their homologous fides, we have CPD = $\frac{4}{9}$CAE, alfo, fince AP = $\frac{1}{2}$PC, and the triangles APD, CPD, are between the fame parallels, APD = $\frac{1}{2}$CPD = $\frac{2}{9}$CAE: therefore ACD, the fum of thefe, = $\frac{6}{9}$($\frac{2}{3}$) CAE, or $\frac{1}{3}$ABC, the whole triangle being bifected by the line CE. In like manner, fince CK = $\frac{2}{3}$BC, it might be fhewn that the triangle CDB = $\frac{1}{3}$ABC; and confequently ADB, the remaining triangle, = $\frac{1}{3}$ABC; and the three triangles formed by lines drawn from D to the angular points, are equal to each other.

To determine the point D by CALCULATION, firft find the area of ABC, which as performed by a well known rule, is 299·33259094: this divided by $\frac{1}{2}$CA gives 19·955506 = BH. Alfo ($\frac{1}{3}$ area ÷ by $\frac{1}{2}$CA) = 6·651835 = DG; ($\frac{1}{3}$ area ÷ by $\frac{1}{2}$AB) = 8·314794 = DI; ($\frac{1}{3}$ area ÷ by $\frac{1}{2}$BC) = 7·675194 = DL. Then, by reafon of the parallels DP, BA, — DG, BH, we have, BH : DG (: : 3 : 1) : : BA : DP = 8. Again $\sqrt{(DP^2 - DG^2)}$ = 4·44444497 = GP; wherefore, GP + AP = AG = 14·444445 nearly. And $\sqrt{(GA^2 + GD^2)}$ = AD = 15·90248097, by which we have determined D, fince AP and DP are known. The diftances BD and CD may be found by means of 47, Euc. I. AB, BC, DI and DL, as well as AD, being known; as may be readily traced out, by thofe who have leifure and inclination to purfue the operation through all the requifite fteps.

N. B. To avoid the imputation of plagiarifm it may be proper to remark that the above conftruction for the point D, is nearly fimilar to one given in No. 11, of the Scientific Recepticle; but as the truth of the conftruction is not demonftrated in that work, I have here fupplied the deficiency.

The fame, anfwered by Merones Minor.

Fig. 510, Pl. 27. Bifect the ∠s A, B, C, of the given △ by the lines AE, BF, CD, and their interfection, O, will be the place of the weight. Then whether the weight be fupported at A, B, and C, or only at A and E, the preffure at A will be the fame;

same; and which, by Em. Mech. 31, Cor. 2. Bon, is $= \frac{100EO}{AE}$

$$= \frac{100BE}{AB+BE} = \frac{100BC}{AB+BC+AC},$$ *ibid. Geom.* 25, and and 17. *Proportion*, = 30lbs. In like manner, the pressure at B =

$$\frac{100AC}{AB+BC+AC} = 42\tfrac{1}{4}\text{lb}, \text{ and that at } C = \frac{100AB}{AB+BC+CA}$$

$= 27\frac{3}{4}$lb.

He also observes that in the second case, as the pressures are to be equal, the distances must be equal also; so that the centre of the circumscribing circle will be the point required, and its distance from each of the angular points = 15·5744.

This question was ingeniously answered by Messrs. Barron, Buckham, Davis, Hill, May, *and* Lowry.

XV. QUESTION 225, *answered by Mr.* Cunliffe.

Multiplying the terms of the series by $\dot{x}$ and taking the fluents, there will be had,

$h.l.(1+x)+h.l.(1+x^2)+h.l.(1+x^4)+h.l.(1+x^8)$ &c. *ad infinitum*: put = h. l. (y); then by the nature of logarithms, $(1+x)\times(1+x^2)\times(1+x^4)\times(1+x^8)$ &c. $= y$; this last expression being actually multiplied becomes

$1+x+x^2+x^3+x^4$ &c. $= y$. But it is well known that

$1+x+x^2+x^3+x^4$ &c. $= \frac{1}{1-x}$, and therefore

$(1+x)\times(1+x^2)\times(1+x^4)\times(1+x^8)$ &c. $= \frac{1}{1-x}$. In which

expression, writing x^{2^n} in the place of x it becomes

$$(1+x^{2^n})\times(1+x^{2^{n+1}})\times(1+x^{2^{n+2}})\times(1+x^{2^{n+3}})\text{\&c.} = \frac{1}{1-x^{2^n}}:$$

but $1+x^{2^n}$, is the $(n+1)$th factor of the foregoing continued product; therefore dividing the former expression by the latter there will be had

$(1+x)\times(1+x^2)\times(1+x^4)\times(1+x^8)$ &c. to n factors =

$1-x$

$\frac{1-x^{2^n}}{1-x}$, whence by the property of logarithms,

h. l. $(1+x)$ + h. l. $(1+x^2)$ h. l. $(1+x^4)$ &c. to n terms =

h. l. $(1-x^{2^n})$ — h. l. $(1-x)$.

Taking the fluxions of this expreſſion and dividing by $\dot{x}$ we get

$$\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8} \text{ \&c. to } n \text{ terms} = \frac{1}{1-x} - \frac{2^n \times x^{2^n-1}}{1-x^{2^n}}.$$

And thus the queſtion was anſwered by Mr. Whalley. *Other anſwers were received from Meſſrs.* Gregory, Hill, Lowry, May, Peacock, *and* Thornoby.

XVI. QUESTION 226, *anſwered by Mr.* Gregory, *the propoſer.*

In order to ſolve this queſtion, let the radius of either wheel be denoted by r, its weight by w, the radius of either axle by d, and the weight ſuſpend from it by p: then, according to Prob. 15, pa. 40. *Bank's Treatiſe on Mills*, we have $\frac{dpr}{r^2w+d^2p}$ for the accelerating force at the circumference, that of gravity being 1; this being drawn into $16\frac{1}{12}$ produces ·2007280288 for the velocity of the circumference of the wheel: and, as $r:d$, or as $2:\frac{1}{4}::$ ·2007280288 : ·0250910036, velocity of the circumference of the firſt axle, or ſpace paſſed over by the weight ſuſpended from it in the firſt ſecond; let this be denoted by a. Again, in the ſecond wheel, $\frac{\cdot 25\times 5\times 1\cdot 75}{(3\cdot 0625\times 40)+(\cdot 0625\times 5)} =$ ·0178131543 is the accelerating force at the circumference; which drawn into $16\frac{1}{12}$ gives ·2864948984, velocity of the circumference of the wheel; alſo, as $r:d$, or as $1\frac{3}{4}:\frac{1}{4}::$ ·2864948984 : ·0409278426 velocity of the circumference of the ſecond axle, or ſpace paſſed over by the weight ſuſpended from it in the firſt ſecond; let this be repreſented by b. Then, putting x for the ſpace deſcended by the firſt weight in the whole time, by the nature of deſcending bodies, we have $a:x::1^2:\frac{x}{a}$, ſquare of the whole time; and, by the

queſtion

[illegible] $b : x + 50 :: t^2 : \frac{x + 50}{b}$, square of the whole time [illegible]; therefore, $\frac{x}{a} = \frac{x + 50}{b}$, whence $x = \frac{50a}{b - a} =$ [illegible] $=$ [illegible] feet. Consequently, ·02509100[illegible] : [illegible] :: t^2 : [illegible] the square of the time and [illegible] the time required.

Or the time might be found rather more concisely, by saying as ·0138368 : (the difference between the distances described by the weights in the first second : (the given difference) :: [illegible] which gives the time of description as before.— Other methods of calculation give 56·19 for the result, corresponding nearly with the above.

The same, answered by Miss Susan May.

Put W = weight of the greater wheel, = 50lb. and R = radius = 2 ft.

w = weight of the lesser wheel, = 40lb, and r = radius = 1½ ft.

a = rad. of the axle = ½ ft. and b = weight suspended therefrom = 5lb.

g = 16$\frac{1}{12}$ feet, and t = time required.

Then, because the weight of the wheels is supposed to be [illegible] [illegible] into their circumferences it follows from Prop. XIX. S[illegible] VI. of *Atwood's Treatise on Motion* that the accelerative force [illegible] the circumference of the axle of the greater wheel is

$\frac{a^2b}{WR^2 + a^2b} = \frac{1}{161} (= p)$, and that the accelerative force at

circumference of the axle of the less wheel is $= \frac{a^2b}{wr^2 + a^2b}$ [illegible]

$\frac{1}{73} (= q)$. And by Prop. IV. Sec. III. Cor. 5, *ibid*, the sp[ace] descended by the weight attached to the greater wheel is $= g$[illegible] and the space descended by the other weight is $= gqt^2$. Therefore

$g(q - p)t^2 = 50$, or $t = \sqrt{\frac{50}{g(q - p)}} = 56·19$ nearly, for time required.

[illegible] *solutions were also received from* Messrs. Bosw[ell], Hill, Lowry, *and* Meroaes Minor.

XV

XVII. QUESTION 227, *answered by Mr.* Cunliffe, *the proposer.*

Let ACB represent the triangular plot of land, with the perpendiculars DD, BF, and AE as per figure 512, plate 27.

Put $AC=a$, $AD=c$, $CB=b$, and $BD=d$. Then there will be had

$a^2-c^2=b^2-d^2=CD^2$, or $(a+c)\times(a-c)=r\times(b+d)\times(b-d)\div r=CD^2$;

put $a+c=r\times(b\times d)$ then $a-c=(b-d)\div r$; from these equations

$$a=\frac{r^2\times(b+d)+d-b}{2r}, \text{ and } c=\frac{r^2\times(b+d)+b-d}{2r}.$$

Now $b^2-d^2=CD^2$, must be a rational square; therefore put $r\times(m^2+n^2)=b$, and $r\times(m^2-n^2)=d$, then will

$$a=\frac{r^2\times(b+d)+d-b}{2r}=r^2m^2+n^2\text{; and }c=\frac{r^2\times(b+d)+b-d}{2r}=r^2m^2-n^2$$

from whence $AB=AD+DB=c+d=(r+1)\times(rm^2-n^2,)$ where r, m, and n, may be taken at pleasure.

Therefore take $r=2$, $m=3$, and $n=2$: then $CB=r\times(m^2+n^2)=26$, $BD=r\times(m^2-n^2)=10$, $AC=r^2m^2+n^2=40$, and $AB=(r+1)\times(rm^2-n^2)=42$: but as each of these is divisible by 2, we may take $CB=13$, $BD=5$, $AC=20$, and $AB=21$.

Then by the similar triangles ACD, and ABF

$AC:AD::AB:AF=(AD\times AB)\div AC=(16\times 21)\div 20=16\frac{4}{5}$,

whence $AC-AF=CF=3\frac{1}{5}$. Also by the sim. $\triangle$s CBF, CAE, $CB:CF::CA:CE=(CF\times CA)\div CB=64\div 13$.

Wherefore, reducing the numerical values of AC, BC, FC, and EC to a common denominator, there will be had

$$AC=\frac{20.\ 13.\ 5}{13\ 5}=\frac{1300}{65},\ BC=\frac{13.\ 13.\ 5}{13.\ 5}=\frac{845}{65}, \text{ and } AB=\frac{21.\ 13.\ 5}{13.\ 5}=\frac{1365}{65}.$$

Therefore rejecting the common denominator 65, there may be taken $AC=1300$, $BC=845$, and $AB=1365$; and these are the numerical values of the sides of a triangle whose perpendiculars and segments of the sides made thereby are all whole numbers. The above numbers seem to be the least for an acute angled triangle, or one whose perpendiculars fall within it: but if we admit

mit an obtuse angled triangle the side AB′ will be 715, and found by taking the segment DB′ from D towards A.

These are the least numbers of the kind that I have hither found; however they may not be the least, as perhaps a different process might discover others that are less. An isosceles triangle of less dimensions might readily have been found that would answer; but that is not what was meant.

Messrs Gregory, Lowry, *and Miss* May *also favoured us with neat solutions to this question.*

XVIII. QUESTION 228, *answered by Mr.* John Andrew, *Cark.*

ANALYSIS. Fig. 513. Pl. 27. Suppose the thing done, B and H the given points, CD the line given in position, CHI the given angle, and BCHI the circle required, the centre of which is O. Join the given points B, H, with the straight line BH, which bisect at A by the ⊥ AG, meeting CD in G, which is a given point, and AG a given line. From A and O, upon CD drop the ⊥s AP, OP′, and draw the radii CO, HO and IO, parallel to which draw AD.

The ∠ COI = 2 CHI (Euc. 20. III.), theref. the ∠ COP′ = P′OI = CHI = PAD, wheref. PAD is = the given ∠, and AD a given line. Moreover the right ∠d △s IP′O, DPA are equiangular, and so are the △s GOI, GAD; hence AG : AD :: GO : OI = OH : but AG and AD are given lines, theref. the ratio of OG : OH becomes known. Now, G and H being given points, the centre O may be found by the following

CONSTRUCTION. Having found the points A, G, and D, as in the *Analysis*, join GH, and from A to GH produced, apply AQ=AD; also draw HO parallel to the line joining the points A and Q, to meet AG at O, the centre of the inscribed circle.

The same, answered by Mr. Lowry.

CONSTRUCTION. Fig. 514. Pl. 27. Let A and B be the given points, and CD the straight line given by position. Bisect the line adjoining the points A, B, at F, and draw FO ⊥ to AB meeting CD in O. On CD demit the ⊥ FH, and draw the FS to make the ∠ HFS = to the given ∠, and meet CD at S. In a semicircle described on the diameter AO, apply OQ = to a 4th proportional to SF, OF and AF. Draw AQ to meet FO at G; so shall G be the centre of the required circle.

Join

Join A, G; G, D; A, D; and A, C; and on CD demit the ⊥ GI. It is evident that the △s AFG, GOQ are equiangular; theref. AG = GD : AF :: GO : OQ, but by construction,

SF : OF :: AF : OQ; therefore, by equality,

GD : GO :: SF : OF; wherefore GD is parallel to SF.

Consequently the ∠ CAD = DGI = SFH = the given ∠ by const.

The same, answered by Mr. I. H. Swale.

Fig. 515, Pl. 27. Suppose the circle described through the given points A, B, as required, and cutting the indefinite right line CD in the points C, D. Join BA, BC, BD; draw DE ∥ to BA meeting the circle at E, and join CE.

Then, since the ∠ DBC = DEC, and ∠ EDC = BPC, are given, the ∠ DCE will also be given. Bisect AB at G, by the ⊥ FGH, meeting CD, DE in F and H, then will DH = HE; therefore DF : FC is a given ratio; for, from above, the △ DCE is given in species. Consequently, since, the rect. CP·PD = AP·PB, and the line FP are given we have the following

Construction. Let the right line BA, joining the given points B and A, meet the indefinite right line CD, given in position, in P. Bisect BA in G and draw GF ⊥ to BA to meet CD at F; draw FK to BA making the ∠ FKP = the given one, and bisect PK at L. Draw PQ ⊥ to CD and = to the side of a square = to the rect. BP·PA; join QF; make FI : FQ = LG : LP (LK), and on QI describe a semicircle cutting CD in C; then through A, B and C, describe a circle, and it is done.

Demonstration. Let the circle meet CD again in D; join BC, BD; IC, QC, QD; draw DE ∥ to BA to meet the circle at E, join CE, produce FG to meet DE at H, and let LR, drawn ∥ to GF, meet PF at R.

By constr. PQ^2 = PB · PA = PC × PD, and PQ ⊥ to CD; therefore the points D, Q, C are in a semicircle, or CQ ⊥ to QD. But by constr. the points I, C, Q, are in a semicircle, or CQ ⊥ to CI. theref. CI, DQ are parallel; and therefore

FC : FD ± FI : FQ = LG : LB, by constr. = RF : RP, by ∥s; also DH : HE = PL : LK, *i.e.* DC : PF = CE : FK, or CE is ∥ to FK. Hence, ∠ PKF = DEC = DBC by the circle = the given ∠ by constr.

The same, answered by Mr. James Whalley.

Analysis. Fig. 516, Pl. 27. Suppose the circumference of the circle to pass through the two given points E and A, and to cut the line FG given by position in the points F, and G,

so that drawing FE, GE, the ∠ FEG may be of a given magnitude. Draw the right line EA cutting FG in I, and suppose GO to be a tangent to the circle at G meeting EA produced in O. Then, by Euc. 36. III. $OA \times OE = OG^2$, and, by Euc. 32. III. the ∠ IGO=FEG = a given ∠; also the ∠ GIO is given from the position of FG, whence the △ IGO is given in species, and theref. the ratio of OG to OI, or of $OG^2 = OA \times OE$ to OI^2, is given, Now, E, I, and A are given points in the indefinite right line EO, theref. the point O may be found by Prob. 6, of *Wales's Determinate Section*, and belongs to fig. 29, of that problem.

Ingenious Solutions to this question were also received from Messrs. Cunliffe, Hill, May *and* Peacock.

XIX. QUESTION 229, *answered by the proposer, Mr.* Robert Wallace.

Let EBHCGF (fig. 541, pl. 28.) be a paraboloid, and A a corpuscle of matter situated in the axis produced. Put AD = y, AH = a, AB = z, $y-a$ = HD, theref. $y^2 + DC^2 = AC^2 = z^2$, $b(y-a) = DC^2$ (b, being the latus rectum), that is $by-ab=z^2-y^2$, theref. $y^2+by=z^2+ab$, and theref. $y^2+by+\frac{1}{4}b^2 = z^2+ab+\frac{1}{4}b^2$; hence $y = \pm\sqrt{(z^2+ab+\frac{1}{4}b^2,)}$

$-\frac{1}{2}b$, $\dot{y} = \dfrac{z\dot{z}}{\sqrt{(z^2+ab+\frac{1}{4}b^2)}}$: put $ab + \frac{1}{4}b^2 = g^2$

then $\dot{y} = \dfrac{z\dot{z}}{\sqrt{(z^2+g^2)}}$.

Now the attraction of all the particles in the circular surface BDC (vid. Simpson's Fluxions, Vol. 2d. Art. 376,) is = $(AD \times AB^{n+1} - AD^{n+2}) \times [2P \div (n+1)]$, P, being = 3·14159, &c. Therefore by substituting and fluxing

we obtain $(y\dot{y}z^{n+1} - y^{n+2}\dot{y}) \times [2P \div (n+1)]$

$= [(\sqrt{z^2+g^2} - \frac{1}{2}b) \times \dfrac{z\dot{z}}{\sqrt{(z^2+g^2)}} \times z^{n+1} - y^{n+2}\dot{y}] \times \dfrac{2P}{n+1}$

$= [(z\dot{z} - \dfrac{bz\dot{z}}{2\sqrt{(z^2+g^2)}}) \times z^{n+1} - y^{n+2}\dot{y}] \times \dfrac{2P}{n+1}$

= (z

$$=(z^{n+2}\dot{z}-\frac{bz^{n+2}\dot{z}}{2\sqrt{(^{2}z-g^{2})}}-y^{n+2}\dot{y})\times\frac{2P}{n+1}=$$ the fluxion of

the whole force, the whole fluent of which, being found by *Simpson's first volume,* and corrected will give the whole force of the attraction of the paraboloid, &c.

XX. QUESTION 230, *answered by the proposer, Mr.* Cunliffe.

Denote the required fractions by $\frac{\frac{1}{4}x}{x+y}$ and $\frac{\frac{1}{4}y}{x+y}$; for upon trial it will be found that if either of them be added to the square of the other, the sum thence arising will be a rational square. Therefore it only remains to determine such values for x and y as shall make the sum of the squares and the sum of the cubes of the above fractions rational squares,

viz. $\frac{1}{16}\times\frac{x^2+y^2}{(x+y)^2}$ and $\frac{1}{64}\times\frac{x^3+y^3}{(x+y)^3}=\frac{1}{64}\times\frac{x^2-xy+y^2}{(x+y)^2}$ each a square; or which amounts to the same thing, x^2+y^2 and x^2-xy+y^2, must each be a square.

Put $2mn=x$ and $m^2-n^2=y$, then $x^2+y^2=(m^2+n^2)^2$, which is manifestly a square: also $x^2-xy+y^2=m^4-2m^3n+2m^2n^2+2mn^3+n^4=$ a square. Assume m^2-mn+n^2 for its root, and then $m^4-2m^3n+2m^2n^2+2mn^3+n^4=(m^2-mn+n^2)^2=m^4-2m^3n+3m^2n^2-2mn^3+n^4$, which being reduced gives $m=4n$; wherefore $x=2mn=8n^2$, and $y=m^2-n^2=15n^2$; whence $\frac{\frac{1}{4}x}{x+y}=\frac{8}{92}$ and $\frac{\frac{1}{4}y}{x+y}=\frac{15}{92}$, which are two fractions that will answer the conditions of the question.

The same question answered differently.

Let $\frac{1}{4}-x$ and x denote the two required fractions;—for it will be found that either of them being added to the square of the other will make a rational square.

It therefore only remains to find the value of x, so that the sum of the squares, viz. $\frac{1}{16}-\frac{x}{2}+2x^2$, and the sum of the cubes,

viz. $\frac{1}{64} - \frac{3x}{16} + \frac{3x^2}{64}$, of the two fractions may both be square numbers; or, which is the same thing, that $1 - 8x + 32x^2$ and $1 - 12x + 48x^2$, may each be a square number. Assume $4nx - 1$ for the root of the former of these, that is, put $1 - 8x + 32x^2 = (4nx - 1)^2 = 16n^2x^2 - 8nx + 1$, which gives $x = \frac{n-1}{2(n^2-2)}$; this being written for x in the other expression $1 - 12x + 48x^2$, it becomes $1 - \frac{6(n-1)}{n^2-2} + \frac{12(n-1)^2}{(n^2-2)^2} = \frac{1}{(n^2-2)^2} \times (n^4 - 6n^3 + 14n^2 - 12n + 4)$, which must be a square number by the question, therefore $n^4 - 6n^3 + 14n^2 - 12n + 4$ must be a square. Assume $n^2 + 3n - 2$ for its root, or make $n^4 - 6n^3 + 14n^2 - 12n + 4 = (n^2 + 3n - 2)^2 = n^4 + 6n^3 + 5n^2 - 12n + 4$, which being reduced gives $n = \frac{3}{4}$; therefore $x = \frac{n-1}{2(n^2-2)} = \frac{1-n}{2(2-n^2)} = \frac{8}{92}$, and $\frac{1}{4} - x = \frac{1}{4} - \frac{8}{92} = \frac{15}{92}$, which are the very fractions obtained by the other method of solution.

Mr. Lowry *and Miss* May *likewise sent solutions to this question.*

XXI. QUESTION 231, *answered by Mr.* James Whalley.

Fig. 117, Pl. 27. *Analysis.* Let ABCD represent the trapez. AB the given side, P the given point therein, through which the side CD, opposite to the given side AB, passes; CAB the given ∠, and QE the right line given by position, in which is situated the angular point D opposite to the given ∠ A. Produce AC to meet QE in E, and draw BF ∥ to QE meeting AE in F and join FD, DB.

The area of the △ FEB = △ FDB, by Euc. 37. I. theref. △ AEB = trap. AFDB = a given space. But the area of the trap. ABCD is given by the question. Wherefore the area of the △ CFD becomes known. Whence the following

CONSTRUCTION *by Mr.* James Cunliffe.

Draw the right line PF, and produce AF, so that the area of the △ FPS may be = to the given space CFD; through S draw an indefinite right line SN parallel to PF. Then through P draw

draw the right line NPCD, *by Prob.* 37, *Simpson's Geometry*, so that PN = CD, and the thing is done.

The same, answered by Mr. I. H. Swale.

Fig. 517, Pl. 27. Suppose the trap. ABCD determined as required, AB the given side, BAC the given ∠, P the given point in BA produced, DC the side passing through P, and DE the right line given in position, in which the ∠ BDC, or the point D is posited: Produce AC to meet DE at E and join EB meeting PD in Q′, then will the point E be given, therefore the △ AEB will be given in magnitude: but the trap. ABCD is also given, theref. their difference, or the difference of the △s CQ′E, DQ′B will be given; theref. by adding common space DQ′E to each, the difference between the areas of the △s DEC, DEB will be given; and, since the base DE is common to those △s, the rectangle under DE, and the difference of the ⊥s from C and B upon DE, will be given: Now, draw CG, BF ⊥ to DE, and BF ∥ to FD, meeting CG in H. Then, since BF = GH, it follows that CH × DE is a given rectangle. Draw BE′ ⊥ to AE, the △s BE′F, CHF are similar, theref. CF : CH = BE′ : BF, a given ratio, wheref. the rect. CF × ED is given. Now E and F are given points, and ED, EA straight lines given by position, and P being also a given point, the problem is, therefore, reduced to the *sectione spatii* of the Ancients.

The same, answered by Mr. Lowry.

Fig. 518, Pl. 27. CONSTRUCTION. Let AB be the given side, P the given point, and ZL the straight line given by position. Draw AC to make the given ∠ with AB, and on AB constitute a parallelogram BAGK = to twice the given area. By Prob. 37 of *Simpson's Geometry*, draw PI to meet AC in C, ZL in D, and GK produced in I, so that CD may have to CI the given ratio of AB to PB. Join BD, and ABCD will be the trapezium required.

Through C and D let CE, VDF, be drawn ⊥ to AB, and ∥ to AB, let CQ be drawn, meeting DF in Q.

Then by reason of the parallels,

QD : QV :: CD : CI :: AB : PB, by construction; therefore, QD·PB = QV· AB, and, by adding CE·AB to each, we have QD·BP + CE·AB = (QV + CE)·AB = the paral. BG. Again, PB·DF = PB·(EC + QD) = 2 △ PDB, and

PA·CE = CE·(PB—AB) = 2 △ PAC; therefore, their difference QD·BP + CE·AB = trap. ABCD = parallelogram BAGK.

Consequently the trapezium is equal to the area. The rest is evident from the construction.

[illegible] QUESTION [illegible] answered by Mr. Lowry.

[illegible] inscribed in [illegible] ence of the [illegible] AB, DC, demit the [illegible] meet EF in G and [illegible]; and CF.

[illegible] 19. VI.)

[illegible] KDF, CEF, that [illegible] that [illegible] CKF, EID, that [illegible]

[illegible] of the fides AB, BC, and [illegible] AD, DC, theref. [illegible] excesses of [illegible] AB, BC, DC and AD [illegible] LB + KC, [illegible]

[illegible] that the [illegible] under the [illegible] and rect- [illegible] the [illegible]

[illegible] for finding the area [illegible] more commodious [illegible]

[illegible] QUESTION [illegible] answered by Mr. C[illegible].

[illegible] ACB [illegible] a circle, [illegible] Q and R. Draw [illegible] and through the centre O [illegible] CI in P;— through P [illegible] AB, to meet the sides AC, BC in [illegible] and [illegible] OQ, OR.

Since

Since ab is parallel to AB it will be bisected by CI in P, or $aP = Pb$. Now TP is $\perp$, or at right angles, to AB, by Euc. 18. III; therefore PO is $\perp$ to ab, by Euc. 28. I.; wheref. the $\triangle$ aOb is isosceles, or $aO = Ob$; but OR = OQ, and $\angle$ $OQa = ORb$ = a right $\angle$, by Euc. 18. III.; theref. $\angle$ $bOR = aOQ$, by Euc. 4. I.

Because $\angle$ $OPb = ORb$ = a right $\angle$, a circle will pass through the points O, P, b, R, and therefore $\angle$ $bOR = bPR$, by Euc. 21. III. For the same reason a circle will pass through the points O, P, Q, a, and therefore $\angle$ $aOP = aPQ$, consequently $\angle$ $bPR = \angle$ aPQ; therefore the points Q, P, R are in the same straight line, by Euc. 15. I.

The same, answered by Mr. Lowry.

Fig. 520, Pl. 28. Let ACB be the $\triangle$, CI the line bisecting the base, O the centre of the inscribed circle, QP a part of its diameter drawn from the point where it touches the base; and R and Q the other points of contact. About the $\triangle$ describe a circle, and let the diameter XIE be drawn $\perp$ to the base; also let CE be joined, which will pass through O, the centre of the inscribed circle.....Now, if the line joining the points R and Q meets OP at P, the truth of the proposition will appear manifest if we can prove the points I, P, C to be in a straight line. In order to do this, through I, draw GIH $\parallel$ to RQ to meet the sides at G and H, also draw AY $\parallel$ to BC to meet IG produced at Y, and join GE. Then because RC = CQ, CG will be = to GH, also AY is = AG = BH; theref. AG is = half the diff. of the sides AC, CB; wheref. by a well known property, GE is $\perp$ to AC, and consequently $\parallel$ to PO. Theref. the $\triangle$s EGI, ROP are equi-angular, and

GI : GE : : RP : RO. also CG : GE : : CR : RO; wherefore,

GI : CG : : RP : CR. Therefore I, P, C, are in a straight line.

If O′ be the centre of a circle touching the base and the continuation of the other two sides in X′, R′, and Q′, then if O′X′ be drawn to meet IC produced in P′; the points R′, P′, Q′, will be in the same straight line.

The same, answered by Mr. Swale.

Fig. 520, Pl. 28. Let ACB be a plane $\triangle$, bisect the base AB at I by the $\perp$ IX; also bisect the vertical $\angle$ ACB by the line CE, meeting IE in E: demit upon the sides CA, CB, the $\perp$s EG, EH, and join GH. Then, I say GH will pass through I:—Fo

I:—For, let GH meet EC in D, then since EC = EC, ∠ ECG = FCH, and ∠ EGC = EHC, the △s EGC, EHC are equal in all respects, theref. CG = CH and theref. the △ GCH is an isosceles one. But CD or CE bisects the vertical ∠ GCH of the isosceles △ GCH, theref. GD is ⊥ to CE. Hence, by Harrison's Propositions in the Repository, GD passes through I, the middle of AB.

Now, take any other point O in EC, and demit upon CA, CB the ⊥s OR, OQ: draw OP ∥ to FI, meeting CI in P, and join RQ. Then, I say, RQ will pass through P.—For, let RQ meet OC in D′, and through P, ∥ to AB draw *ab* meeting CA, CB in *a* and *b*; then since AI = IB, and *ab* ∥ to AB, the base *ab* will also be bisected by the line CI, that is, *a*P = P*b*. Now, by reasoning as above, it appears that QD′ is ⊥ to CO, and also passes through P the middle of *ab*, therefore RQ passes through P.

Whence, taking the point O, the centre of the inscribed circle, the demonstration follows from the foregoing general one.

Cor. Produce PO to meet AB at T, produce also IE to H′, making EH′ = EG = EH: the points C, T, H′ are in a right line.

XXIV. QUESTION 234, *answered by Mr.* I. H. Swale.

Fig. 181, Pl. 28. Produce FG, KH, to meet at Q, and join QA, QB, QC; let AL, BQ, CQ meet BC, CF, BK at I, S, R, respectively: the lines BK, CF intersecting at P.

Now, QG = AH = AC, GA = AB, and the ∠ AGQ = BAC, theref. the △s ABC, AGQ are equal in all respects, and theref. the ∠ GQA = BCA = CAI, that is, because GQ, IA are parallel, QAI is a continued right line, that is, IA produced passes through Q; theref. QI is ⊥ to BC.

Again, since, AC = CK, AQ = BC, ∠ BCA = QAH, [illegible] CAH right angles, it follows that the △s BCK, CAQ are equal in all respects, theref. ∠ CBR = CQI, that is [illegible] a right angle; theref. BR is ⊥ to QC.

In like manner it may be shewn that CS is ⊥ to QB. Hence, QI [illegible] respectively ⊥ to the sides BC, CQ, BQ of the △ [illegible] ⊥s from the respective ∠s of any plane [illegible] sides, are known to meet in the same [illegible] K, CF, intersect each other at the same

The

The same, answered by Merones Minor.

Instead of AL being drawn ⊥ to BC, let it pass through the intersection P of BK, CF. Then AD appears to be ⊥ to CS, for ∠ CFB = BAD, *Emerson's Geom.* 6. II. and ∠ BcF = AcS'. *Ibid.* 2. I; consequently ∠ AS'c = cBF = a right angle. In like manner it may be shewn that AE is ⊥ to BK. But, *ibid.* 24, II. The three ⊥s drawn from A, E, D, upon the opposite sides, all meet in one point M; theref. CP, BP being = and parallel to EM, DM, respectively, they must also meet AL in one point P, when AL is ⊥ to DE or BC.

The same, answered by Mr. Lowry.

The lines mentioned in the question do meet in the same point, as will be manifest from what follows. Let the lines CF, BK, meet in P, and through P draw APL to meet DE at L. Then it must be proved that AL is ⊥ to DE. Draw AD, AE meeting BK in R' and S', and CF in V and Q', and join the points R', V and S'Q'. It is evident from the proposition mentioned in the question that the ∠s BCS', BDS' are equal, and also the ∠s CEQ', CBQ', theref. the ∠s DS'C, BQ'E are right angles, and the points C, E, D, B, S', Q', are in the circumf. of a circle, and conseq. the ∠s Q' S' V, Q'BC, are equal. Again, because of the right angles at S' and Q', the points R', S', Q', V are in the circumf. of a circle, wheref. the ∠ K'R'V is = to the ∠ Q'S'N, that is = to the ∠ Q'BC; theref. R'V is ∥ to BC. Let AL meet R'V at N, then the ∠s R'VS', KQ'S, and PAS' are equal (for the points S', P, Q', A are in the circumf. of a circle), theref. the Δs AS'P, PNV are equiangular, and AS'P is a right angle, theref. PNV is also a right angle, and consequently AN is ⊥ to R'N, and AL to DE.

Messrs. Hill, Marrat, May, *and* Peacock *favoured us with neat demonstrations of this proposition.*

XXV. QUESTION 235, *answered by Mr.* Cunliffe.

When the given point is within the circle.

Analysis. Fig. 522, Pl. 28. Let DPE be the required line, passing through the given point P and cutting the periphery of the circle in the points D and E. Through P draw the diameter AB, upon which let fall the perpendiculars D*m*, E*n*; and from the centre C draw CQ ⊥ to DE. Draw E*e*, QR ∥ to AB, meeting

[illegible] $Cr \perp$ to Dn [illegible]

CASE I. [illegible] $Dm - En$ = [illegible].

[illegible] CP.CQ::DE:De, [illegible] DE = CP × De: now CP is [illegible] $De = Dm - En$ [illegible] the question; from [illegible] CQP being [illegible] diameter CP [illegible] CQ determined [illegible].

CASE II. When $Dm - En$ = a given magnitude.

[illegible] III. [illegible] theref. because [illegible] R [illegible] theref. Rn = Cr = [illegible] Dm [illegible] of Dm and En. Whence [illegible].

CASE III. When the rectangle $Dm \times En$ = a given magnitude.

Draw PT ⊥ to AB, to meet the periphery of the circle in T. The ∠s PDm, and PEn are similar, theref.

$PD : PE :: PD^2 : PD \times PE = PT^2 :: Dm : En :: Dm^2 : Dm \times En$; from whence, by alternation, $PD^2 : Dm^2 :: PT^2 : Dm \times En$.

The △s PDm and PCQ are also similar, wherefore $PD^2 : Dm^2 :: PC^2 : CQ^2$, conseq. $PT^2 : Dm \times En :: PC^2 : CQ^2$; now PT and PC are each given, and the rectangle $Dm \times En$ is given by the question; from whence, therefore, CQ becomes known and a construction thereof is quite obvious.

Or a construction to the third case of this part of the problem will be very obvious from what follows.

Draw TV ⊥ to DE; then, by the sim. △s CQP, and PVT, $PT^2 : PV^2 :: PC^2 : CQ^2$, whence, and from what is done before, it appears that $PV^2 = Dm \times En$, which is a curious theorem.

When the given point is without the circle. Fig. 523, Pl. 28.

CASE I. When $Dm + En$ = a given magnitude.

The chord DE is bisected in Q (Euc. 3. III.). Theref. because of the parallels Ee, QR, DR = Re, whence $mR = 2Cr = Dm + en = Dn + En$ = a given magnitude by the question. [illegible] CQP is a right angle, the locus of Q will be a circle [illegible] CP — Theref. let this circle be described, and the following construction offers itself.

Erect Cr ⊥ to AB and = to half the given sum of Dm and En; through r draw rQ [illegible] to AB, to cut the periphery of the circle whose diameter is CP in Q; through P and Q draw PED

cutting

cutting the given circle in E and D, and the thing is done as is evident from the analysis.

CASE II. When $Dm - En =$ a given magnitude.

It is evident that $Dm - En = Dm - me = De$ a given magnitude by the question. Moreover the $\triangle$s PCQ, and EDe are similar, theref. $CP : CQ :: DE : De$, whence (Euc. 16. VI.) $CQ \times DE = CP \times De$: but CP is given, and De is given by the question, theref. CQ may be determined from hence, and this case of the question thereby readily constructed.

CASE III. When the rectangle $Dm \times En =$ a given magnitude.

Draw the tangent PT:—Then by the parallels Dm, En,
$DP:PE::DP^2:DP \times PE = PT^2::Dm:En::Dm^2:Dm \times En$;
from whence by alternation $DP^2 : Dm^2 :: PT^2 : Dm \times En$.

The $\triangle$s PDm, and PCQ are similar, therefore
$DP^2:Dm^2::PC^2:CQ^2$, consequently $PT^2:Dm \times En::PC^2:CQ^2$; but PT and PC are each given, and the rectangle $Dm \times En$ is given by the question; whence CQ becomes known and the construction from thence manifest.

Or, a construction of the third case of this part of the problem may be had from what follows.

On PC take $PT' = PT$, and upon PT' as a diameter describe a circle cutting the line PD in V, and join $T'V$. The $\triangle$s PCQ, and $PT'V$ are similar, theref. $PT'^2 = PT^2 : T'V^2 :: PC^2 : CQ^2$; wheref. from what is done above, it appears that $T'V^2 = Dm \times En$, *which is another curious theorem.*

The same, answered by Mr. Lowry.

When the given point is without the circle. Fig. 524, Pl. 28.

Let P be the given point, BEFA the given circle, and suppose the chord PEF to be drawn as required, namely, so that demitting on the diameter PBA the $\perp$s EH, FK, their sum difference, or rectangle may be given.

1. When the sum is given. Bisect EF at G, and on AB drop the $\perp$ GI; then GI is evidently $=$ to half the sum of EH, FK, theref. GI is given. Now, if O be the centre of the circle, OG being joined will be $\perp$ to EF; theref. the *locus* of the point G will be a semicircle described on PO; consequently if PQ be drawn $\perp$ to PA and $=$ to half the given sum, a parallel to PA, drawn from Q, will intersect the semicircle at G, through which if PG be drawn it will be done.

If E′ be the point where the circles intersect, PE′ will be a tangent to the circle BEF at E′; theref. it is evident that the given sum must not be greater than twice the $\perp$ drawn from E′ to AB.

2. Whe

2. When the difference is given. In AB find the point P', so that BP' : AP' : : PB : PA; then it follows from Lawson's 15th Prop. (demonstrated in the Repository) that if any line whatever as PEF be drawn to meet the circle at E and F, and FP' be drawn to meet the circle again at S, the line joining ES will be ⊥ to AB: conseq. SH will be = to EH, and theref. the difference between FK and SH will be given. Bisect SF at V, and on AB demit the ⊥ VL, then it is evident that VL is = to half the diff. of FK and SH, and is therefore given. Now, because OV is ⊥ to SF, the *locus* of the point V is a semicircle described on OP'. Theref. if in this semicircle VL be applied perpendicularly to P'O, and = the given length, the point V will be determined, and the remaining part of the construction manifest.

The difference will evidently be a maximum when P'O is bisected at L in which case it is = P'O. Or if P'E' be drawn, and OE' joined; by Lawson's 15th Prop. P'B' is ⊥ to AB; theref. $OB^2 = OE'^2 = OP' \times OP$, and $OG^2 = PO \times OI$, (Euc. Cor. 8. VI.); theref. $OE'^2 : OG^2 :: 2 : 1$.

3. When the rectangle is given.

By similar triangles PE : PF : : EH : FK, and therefore, $PE \times PF = PE'^2 = PO \times PP' . PF^2 :: EH \cdot FK : FK^2$; but by sim. Δs, $PF^2 : FK^2 :: PO^2 : OG^2 = PO \times OI$, therefore by equality, $PO \times PP' : PO^2 :: EH \times FK : PO \times OI$, or $PP' \times OI = EH \times FK$ = a given space · but PP' is given, theref. OI is given, and the method of construction is evident.

The rectangle will manifestly be a maximum when OI is so; that is, when it be becomes = to OP', or when EH and FK coincide.

When the given point is within the circle.

If P' be the given point within the circle, and a point P be found without so that BP : AP : : P'B : P'A, the constructions and determinations will be the very same as the preceding.

The same, answered by Mr. I. H. Swale.

I. When the sum of the perpendiculars is given and the given point within the circle Fig. 524.

Let P' be the given point, draw the diameter BP'A, and P'E' ⊥ thereto, meeting the circle at E', and let the tangent E'P meet P'B produced at P. Perpendicular to BA, in a semicircle on OP, apply IG = half the given sum; draw PG meeting the circle at E and F, and through E, P', or F, P', draw the required chord EP'B', or FP'S.

Draw ES, FB', meeting BA in H, K, and join OG: then since ES, FB' are ⊥ to AB, we shall have HE = HS, and KB

$KB = CF$, or $HS + BK = HE + KF =$, because OG is $\perp$ to EF, to $2IG =$ the given sum by construction.

When the given point is without the circle, Mr. Swale's *Construction* is the same as Mr. Lowry's given above.

II. When the difference of the perpendiculars is given and the given point without the circle. Fig. 524.

Draw to the circle, the tangent PE', and, demit upon AB, the diameter drawn through, the $\perp$ $E'P'$. Perpendicular to BA, in a semicircle described on $P'O$, apply $LV =$ half the given diff. and join $P'V$ which produce to the circle at F, and through P, F, draw the required line meeting the circle at E and F.

Produce FP' to the circle at S and join SE, meeting AB in H. Demit upon AB the $\perp$ FK, also join OV. Then since ES will cut AB perpendicularly at H, we shall have $EH = HS$, and $FK - EH = FK - HS$. Now draw through V, parallel to AB, the line RVN, meeting EH, FK in R, N. The points P', V, O, are in a semicircle, theref. OV is $\perp$ to SF, that is, $SV = VF$; theref. by parallels $FN = SR$, $FK = FN + NK = SR + NK = SR + RH = SH + 2HR = SH + 2LV = EH + 2LV$, and $FK - EH = EH + 2LV - EH = 2LV =$ the given diff. by construction.

When the point is within the circle as at P', it will be determined by the const. given above, the demon. being of course the same.

III. When the rectangle of the perpendiculars is given, and the point without the circle. Fig. 524.

Draw PBA, PE', and $E'P'$ as in the last case. Apply in the circle, $\perp$ to AB, $L''L'$ the side of a square $=$ to the given magnitude and to a semicircle on $P'B$, apply $OS' = OL''$; join $P'S'$, produce $P'S'$ to the circle at E, and through PE draw the required line meeting the circle E and F.

Produce EP' to the circle at B', join $B'F$, meeting AB in K; upon AB demit the $\perp$ EH, join OS' which produce both ways to the circle at G' and H. Then, as before, from known properties $B'F$ is $\perp$ to AB, and $FK = KB'$, or $2FK \times 2EH = B'F \times 2EH =$, by sim. $\triangle$s, to $B'E^2 - EF^2$: But since the points P', S', B, are in a semicircle, BS' will be $\perp$ to EB'; or $B'E + EF = 2B'S'$, and $B'E - EF = 2ES$, that is, $2B'S' \times 2ES' = B'E^2 - EF^2 = B'F \times 2EH = 2FK \times 2EC$; and $FK \times EH = B'S' \times S'E =$, by the circle, to $G'S' \times S'H'$: Now, by constr. $OS' = OL''$, and by the circle $OB = OA = OG' = OH'$, or $G'S' = AL''$, and $H'S' = BL''$; that is, $BL'' \times L''A = G'S' \times S'H' = FK \times EH$ from above, $= L''L'^2$, by the circle, $=$ the given magnitude by construction.

When the point is within the circle, the const. will be the same as the above.

Messrs. Andrew, Thornoby, *and* Whalley *also answered this question very ingeniously.*

XXVI. QUESTION 236, *answered by the proposer, Mr.* Lowry.

CONSTRUCTION. Fig. 523, Pl. 28. Let AB, BC and AC be the three straight lines given by position, and let C*a*, C*b*, C*d* be respectively ∥ to the lines drawn from the required point to AB, BC, and AC. Produce *a*C till it meets AB at D, and take D*a* = to the given sum; draw Q*a*Q ∥ to AB to meet AC, BC produced, in S and X. Take C*b* and C*d*, each = to C*a*, and draw S*b*M, X*d*N, to meet BC and AC, in M and N respectively; draw also MH ∥ to C*b*, to meet AC at H, and LN ∥ to C*d* to meet CB at L; join LH meeting AB, produced at R. Through H draw HIW ∥ to C*a*, and continue it to meet LW, drawn ∥ to AB, cut W. Then, by Prob. 2nd, of the Determinate Section by Snellius, at HR at P, so that the rectangle HP·PL, may have to the rectangle contained by PR and the given line, the given ratio of HW·HL to LN·HM; then P is the point required.

DEMONSTRATION. From P draw PDG, PE, and PF ∥ to C*a*, C*d*, and C*b* respectively, and let the line drawn through the points I, L, meet GP at O, also draw LK ∥ to C*a*.

Then, because of the parallels, it is evident, since C*a* (= C*b*) is = C*d*, that LK is = to LN, and HI = HM. And by sim. △s,
IL : IO : : LK : OG (: : LH : HP) : : LN : PE,
but LK is = LN, therefore OG is = PE. Moreover,
IL : OL : : HL : LP : : HM : PF : : HI : PO,
but HI is = HM, therefore PO is = PF. Consequently,
PE + PF + PD = D*a*, that is, = the given sum, by constr.
Again, let Q be the given line. Then by sim. △s,
HP:PE::LH:LN, and PL:PF::LH:HM, and compounding
HP·PL:PE·PF::LH²:LN·HM. But, by construction,
HP·PL:PR·Q::HW·HL:LN·HM, therefore, *ex equo perturbate*
PE·PF:PR·Q::HW·HL:LH²::HW:HL; and by sim. △s,
HW : LH : : DP : PR : : DP·Q : PR·Q; therefore
PE·PF:PR·Q::DP·Q:PR·Q; and theref. PE·PF = DP·Q.

The same, answered by Mr. Cunliffe.

I. To find a point, in one of three right lines, given by [...]ence if two right lines be drawn in given angles to th[...] [...]e sum of the two lines so drawn may be of a give[...]

CON[...]

CONSTRUCTION. Fig. 526, Pl. 28. Let AB, AC and BC be the three lines given by position: Take AM of the given length and making with AB the given angle MAB; through M draw M*ba* parallel to AB, meeting BC and AC in *b* and *a* respectively. Draw AV to meet CB in V, making the $\angle$ AVC equal to that made with the line BC: upon AV take A*m* = AM, and draw *am* cutting BC in G; through G draw GD parallel to AV meeting AC in D, and the thing is done.

DEMONSTRATION. Through D, draw *g*L ∥ to AM, terminating in M*a* and AB in *g* and L. By reason of the parallels D*g*, AM; DG, A*m*; *a*A : AM = A*m* : : *a*D : D*g* = DG; theref. DL + DG = DL + D*g* = L*g* = the given length by constr. and Euc. 34. I. Moreover, $\angle$ DLB = MAB, and $\angle$ DGC = AVC, the given angles by construction and Euc. 28. I.

Q. E. D.

The above being premised, the solution of the problem may be as follows;

Fig. 527, Pl. 28. Let AB, AC and BC be the three lines given by position. In one of them as AC, by the preceding Lemma, find the point D, such, that drawing DG to BC, and DL to AB in the given angles of the straight lines to be drawn to these lines, DL + DG may be equal to the given sum of the three lines. In like manner, in BC find the point E, from whence drawing EH to AC, and EK to AB in the respective given angles of the lines to be drawn to these lines, EH + EK may also be equal to the given sum of the three lines. Through D and E draw the right line DEF to meet AB in F, and this line DEF is the *locus* of the required point P.

For, upon LD and KE take L*g* and K*h*, each equal to the given sum of the three lines, and draw D*h* and *gh*. From any point P in DF draw the lines PQ, PR, PS respectively parallel to DG, EH, DL to meet the lines BC, AC, AB in Q, R and S; also produce SP to meet D*h* and *gh* in *r* and *q* respectively. Because of the parallels PR, EH; PR, E*h*;

DE : DP :: EH = E*h* : PR = P*r*. Also

ED : EP :: *h*D : *hr* :: DG = D*g* : PQ = *rq*; wherefore

PS + PR + PQ = PS + P*r* + *rq* = S*q* = L*g* the given sum by construction.

Now, suppose P to be the point required, upon DF take D*d* equal to the given line, the rectangle under which and PS, shall be equal to the rectangle under PR and PQ, that is D*d* × PS = PR × PQ, and through *d* draw *de* parallel to AB meeting DL in *e*. Then by the sim. △s D*de* and PFS,

D*d* : D*e* :: FP : PS, whence D*d* × PS = D*e* × FP.

Also by reason of the parallels DG, PQ; EH, PR,

DE : DG : : EP : PQ, and DE : EH : : DP : PR, whence

$$=\frac{r^{t+1}-(t+1)r+t}{r^t\times(r-1)^2}.$$ See *Clarke's* Tranflati Lorgna's Series, Form 190, or *Dodfon's* Mathematical Re] Page 65, vol. ii. Whence $t=\log.\left(\frac{(t+1)r-t}{r-p(r-1)^2}\right)\div$] and $t=55{\cdot}228$ the number of quarters required.

And thus nearly was the anfwer given by Merones *and Mr.* Marrat *of Bofton.*

To XXVIII. QUESTION 238, *we have received no anſ*

XXIX. QUESTION 239, *answered by Mr.* Low] *proposer,*

Analysis. Join BP (fig. 528, pl. 28.) and dra make the ∠ BPG = the given one; take PG = PG : GI to make the ∠ PGI = BPE: then, it is evident tha GPI, BPE, are every way equal, theref. GI is = BE PH : PG :: M : N, and make the rectangle N × P = one, and draw HK ∥ to GI meeting AC at L: then, KI M : N, wheref. N × KH = M × GI = M × BE, and I ± N × CF = N × P, or KH ± CF = P = a given Moreover LA and LC are given lines, theref. KL given, and PF may be drawn as in my folution to Quefti It may alfo be drawn differently as follows: Through th K, L, F (fig. 529, pl. 28.) defcribe a circle and draw the TID ⊥ to KF, alfo draw DR and TS ⊥ to LF, and R the line joining the points D, L; draw alfo SI ∥ to D L it is well known that LR is = to half the fum, and half the difference of KL, LF, and that RQ and SI m the middle of KF; we have theref. the following fim *ftruction.*

1. When the fum is given. Take LR = half that draw LD to bifect the ∠ KLF; on LD drop the ⊥ make RD ⊥ to LR; then a femicircle defcribed on the PD will interfect RQ at I.

2. When the difference is given take LS = half the ference, draw ST ⊥ to LS, meeting LT, drawn ⊥ to L and draw SI ∥ to LD; then a femicircle defcribed on the PT will interfect SI at I.

The same, answered by Mr. James Cunliffe.

Case I. When the sum of the rectangles BD × M and CE × N is equal to a given space. Fig. 530, Pl. 28.

Join PB, and draw PF meeting AC in F, and making the ∠ BPF = to the given ∠. In PF take P*b* = PB and divide P*b* in R, so that N : M : : P*b* : PR. Also take RS of such a length that the rectangle RS × N, may be = to the given space, and the ∠ *b*RS = PBA. Then, by *Halley's Apollon. de Sectione Rationis*, from the given point P draw the right line PEG, cutting AC in E and RS in G, so that CE = GS; also draw PD to AB, so that the ∠ EPD = FPB and the thing is done.

For, draw *bd* ∥ to RS, meeting PE in *d*. By the const. ∠ PRS = PBD, Euc. 27. I. = P*bd*, also *b*PB = *d*PD, and taking away the common ∠ *d*PB, there will remain ∠ *b*P*d* = BPD: whence, by Euc. 26. I. the △s *b*P*d* and BPD are equal in every respect; wherefore, by reason of the parallels *bd*, RG; N : M : : P*b* : PR : : *bd* = BD : RG, theref. by Euc. 16, VI. RG × N = BD × M. By the constr. RG + GS = RG + CE = RS; wherefore RG × N + CE × N = BD × M + CE × N = RS × N, the given space, by the construction.

Case II. When the difference of the rectangles BD × M and CE × N is equal to a given space.

Take RS′ on the contrary side of P*b*, and of such a length that the rectangle RS′ × N may be equal to the given space. Then, by *Halley's Apollon. De Sectione Rationis*, draw the right line PEG cutting AC and RS′ in G so that CE = GS′; also draw PD to AB so that ∠ EPD = BPF, and the thing is done.

For, by what is done in the preceding case BD × M = CE × N, and by constr. GS′ — GR = CE — GR = RS′; wheref. CE × N — GR × N = CE × N — BD × M = RS′ × N the given space by the construction.

The same, answered by Mr. I. H. Swale.

Case I. When the sum of the rectangles is given.

Fig, 531, Pl. 28. The lines PD, PE being supposed drawn as required, join BC: Take BF = M, and CG = N, join FC, GB, and draw, to BC, the lines DH, EI, making ∠ BDH = BCF, and ∠ CEI = CBG, and produce DH, EI to meet at Q. The points D, C, F, H, are in a circle, and so are the points E, B, G, I, theref. BF × BD = BC × BH, and CG × CE = CB × CI. Theref. the sum of the rectangles BD × BF, and CE × CG, that is, the sum of the rectangles BD × M and CE × N is = to the sum of the rectangles BC × BH, and BC × CI,

CI, that is, = to the rectangle under BC and the sum of BH, CI, = to a given rectangle: But BC is a given line, therefore the sum of BH, CI is given; theref. HI is a given line. Now, through Q, draw RS ∥ to BC, meeting AB, AC in R and S; draw also RK, SN ∥ to QE, QD, meeting AC, AB, in K and N.

Now, ∠ QHI = BFC, and QIH = CGB, being given, the the △ HQI is given in species, and, being described upon a given base HI, it is therefore given in magnitude. Theref. since HQ : BR = DH : DB, is a given ratio, BR is given, and of course the line RS is given in position.

Moreover, by reason of the parallels, RD : DN = QR : QS = EK : ES, et componendo, RN : SK : = RD : EK, a given ratio; since the points R, S, are given from above, and the lines RK, SN, making given angles with RS, are given by position, or the lines RN, SK, and their ratio therefore given.

Now the lines AR, AK, are given by position, R and K are given points therein, and P is a given point without them; we have only therefore to draw two lines PD, PE, to contain a given ∠, meeting AR, AK, in D and E, and making DR : EK a given ratio, which has been done at question 152.

Case II. When the difference of the rectangles is given.

Fig. 519, Pl. 28. Having drawn lines as in the preceding case, it will appear, that, the rectangle under BC and the diff. of BH, CI is given, and because BC is a given line, the diff. of BH, CI will be given. Bisect BC in M, make IW = HM, and draw MV, WV parallel to DQ, EQ, meeting each other at V, and the lines AB, AC in L and K; through V draw RS ∥ to BC, meeting AB, AC in R, S, and DQ, EQ in U and T...... Then, the diff. of BH, CI will be = to the diff. of MI, MH, that is equal to MW, a given line; theref. the point W is given; conseq. the △ MVW is given in species, magnitude, and position, that is, the point V is given; therefore the line RS passes through a given point and is ∥ to a given line; consequently the lines KS, LN are given in length, N being the point where SN, drawn ∥ to VL, meets AB. Whence, by parallels, DL : LN = VU (= VT) : VS = EK : KS; et alternando, LN : SK = DL : EK = given ratio: But AL, AK, are lines given by position; L, K, given points therein, and P a given point without them, DPE a given ∠, and LD : KE a given ratio. And therefore this case is the same as the preceding one.

Remark. When PD, PE, are to be drawn including a given ang... king the sum or diff. of BD, CE = to a given line, ...usions are still derived.

For,

For, accordingly as the ſum or diff is given, ſet off, above or below C (fig. to Caſe I.) CT or CV equal thereto. Then in the former caſe we ſhall have BD = TE, and in the latter BD = VE: theref. drawing, by the ſolutions to Queſtion 152, the lines PD, PE, making DPE = the given ∠, and BD : TE, or BD : VE in the ratio of equallity, and it will be done. The points B, and T or V being evidently given ones.

XXX. *Or*, PRIZE QUESTION 240, *anſwered by Mr.* Swale.

Let ABCD (fig. 533, pl. 28.) be the circle given in magnitude and poſition, and P, Q, and R the given points.

Suppoſe the required trapezium inſcribed in the given circle, and that the ſides CA, BD; AD, CB, when produced interſect each other at the points P and Q, alſo that one of the diagonals, as CD, paſſes the given point R.

Now, join PQ, and it will be given in length and poſition. From A and B draw AK, BL, parallel to PQ meeting the circle at K and L; then by parallels, the arches AL, BK, are equal. Join AL, and produce LA to meet PQ at G; then, again, by reaſon of the parallels, the ∠ AGP = ALB = ACB, by the circle; therefore the ∠ QGA = ſupp. of the ∠ QCA, or the points Q, C, A, G, are in a circle; therefore, PQ × PG = PC × PA. Again the ∠ PGA = PDA, therefore the points P, G, D, A, are in a circle; theref. GQ × QP = DQ × AQ = BQ × CQ; and therefore PQ × PG + PQ × QG = PQ^2 = PC × PA + QB × BC = the ſum of the ſquares of the tangents drawn to the circle from P and Q *. Wherefore it appears that the given points muſt be ſo ſituated, with reſpect to the circle, that the ſquare of PQ muſt be equal to the ſum of the ſquares of the tangents from P and Q, ſo that the problem *cannot* be conſtructed *generally* for any ſituation of the points, but only for the particular one ſpecified above.

Let O be the centre of the given circle, draw GO meeting the circle in I and H, and CD in S, alſo let CG when drawn meet the circle in F; then, becauſe PQ × PG = PC × PA, and GQ × QP = BQ × QC, it is evident, from Mr. *Harris's* demonſtration to *Lawſon's* 33rd propoſition, that GO is perpendicular to PQ, AK and BL, therefore the arch AK = the arch IK, and the points G, K, B, are therefore in a ſtraight line. Again, becauſe OG is perpendicular to PQ, by *Lawſon's* 11th propoſition IG × GH + GP^2 = AP × PC, that is, = GP × PQ; take

GP^2

GP from each, and IG × GH = PG × GQ; therefore by *Lawson's* 28th proposition IG : GH :: SI : SH; consequently S is a given point: Hence the following simple

CONSTRUCTION. Join the first two given points P, Q; and upon PQ, from the centre O of the given circle, demit the perpendicular OG, meeting the circle in H and I; make HS : SI = HG : GI, and join the third given point R with the point S; let RS meet the circle in C and D, join PD, QD, meeting the circle again at B and A, and join CA, CB: so shall CA, CB, produced pass through the points P and Q, and ABCD will be the trapezium required, as is evident enough from the analysis.

The truth of question 10 in the British Magazine is evident from the expression marked *

The following THEOREM is also true, and may be of use in the construction of Problems.

THEOREM. Let S be the intersection of the diagonals of a trapezium, inscribed in a circle, whose centre is O, join OS, and make OG equal to a third proportional to OS and the radius of the circle: Then the line joining the points of intersection of the opposite sides of the trapezium produced, will pass through the point G and be perpendicular to GO.

The same, answered by Mr. Richard Nicholson, *Liverpool.*

ANALYSIS. Suppose the thing done, ABCD (fig. 540, pl. 28.) the trapezium required, and E, F and P the given points. Draw EF, and CH parallel thereto meeting the circle at H, draw also HAG meeting EF in G, and join GC. Now (Euc. III. 21, and I. 29.) the ∠ ABC = AHC = HGE; theref. a circle will pass through the points A, B, E and G: wheref. (Euc. III. 36.) the rect. EF·EG = EB·EA = EC·ED, a given magnitude; theref. G is a given point, and a circle will pass through the points G, F, C and D; therefore the ∠ HCG = FGC = FDC = CHG, wheref. HG = CG: Draw KI perpendicular to HC, and GI perpendicular to HC also, and HI = IC, theref. GK is one continued straight line, that is, the line GL, drawn through the centre K, will be perpendicular to HC or EF, and therefore the point O becomes known by *Lawson's* 7th Proposition: hence the position of the diagonal AC is given.

CONSTRUCTION. From the centre K draw KG perpendicular to EF, draw any line GH to meet the circle in A and H, HC perpendicular to KG to meet the circle in C and join GC. Through the point of intersection of AC, GK, and the point draw the diagonal AC, complete the trapezium, which inscribed in the given circle when the points are so situated

as

as to render the problem possible*. For the $\angle$ EGA = FGC, and EAG = GFC; therefore EG : GA :: GC : GF or EG·GF = GC·GA = GL·GT; and GL perpendicular to EF, therefore the points A, C, D, and B, are in the circle by Prop. 33, 34, I. *Stewart's Propositiones Geometricæ.*

The same, answered by Mr. James Cunliffe.

Fig. 535, Pl. 28. ANALYSIS. Let ABCD be a trapezium inscribed in a circle, the sides BA, CD being produced to meet in the point E; and the sides AD, BC being produced to meet in the point F. . .Draw the tangents ET, FM, and to the centre O draw EO, FO, TO, MO; also draw Dn to meet FE in n; making the $\angle$ EnD = DAB, and join On.

Then the $\angle$ EnD + DAE = DAB + DAE = 2 right angles; whence the points E, A, D, n, lie in the circumference of the same circle, per Euc. 22, III. Again, since, by construction, the $\angle$ EnD = DAB, the supplements of these angles must be equal, viz. $\angle$ FnD = DCB, consq. the points F, C, D, n, lie in the circumference of the same circle;

wherefore, per Euc. 36, III, $Fn \times FE = FD \times FA = FM^2$,
and $En \times FE = ED \times EC = ET^2$;
and, by addition, $Fn \times FE + En \times FE = FE^2 = FM^2 + ET^2$.

From whence it appears that the points E and F, must always have such a situation with respect to the circle, that the square of the distance between them may be equal to the sum of the squares of two tangents from them to the circle.

The $\angle$s ETO, and FMO are right $\angle$s, per Euc. 18, III; therefore $FO^2 - OM^2 = FO^2 - OT^2 = FM^2 = Fn \times FE$,
and $EO^2 - OT^2 = ET^2 = En \times FE$
and theref. $FO^2 - EO^2 = Fn \times FE - En \times FE = FE \times (Fn - En)$.

Therefore On is perpendicular to FE, as appears from prop. 24, B. II, *Emerson's Geometry.*

Draw MR $\perp$ to FO and join ER; the $\triangle$s FRM and FMO are right angled and similar, theref. FR : FM :: FM : FO, whence $FR \times FO = FM^2$; and it has been shewn that $Fn \times FE = FM^2$; conseq. $FR \times FO = Fn \times FE$, wheref. per Euc. 36, III, the points E, n, R, O, are in the circumference of a circle, and theref. per Euc. 21, III, $\angle$ ERO = FnO = a right angle; wheref. ERM is a straight line and cuts FO at right angles in R. And in the same manner it may be shewn that FT cuts EO at right angles in Q; and theref. it is plain from prop. 34, B. II. *Emer-*

* *Note.* The points must be situated as mentioned in Mr. Swale's solution.

son's

...'s Geom. that the lines FT, EM and O*n* interfect in the fame point I.

Thefe things being premifed the *conftruction* of the problem may be as follows.

Let E and F be the two given points in which the oppofite fides of the trapezium are to interfect, fo fituated with refpect-to the given circle TmM, as to render the queftion poffible; alfo let P be the given point through which one of the diagonals of the trapezium is to pafs. Having found the point I from what is fuggefted in the *Analyfis*, through that and the given point P draw the right line DB cutting the circle in D and B; alfo draw EB cutting the circle again in A, and through I draw AIC to cut the circle in C, then draw the lines BC, CD, DA, and ABCD is the required trapezium.

DEMONSTRATION. Draw the lines AQ, BQ, AO, BO, FD, FD and FC; alfo draw OS $\perp$ to AB. Then from what is deduced in the Analyfis, and Euc. 36, III. $ET^2 = EA \times EB = EQ \times EO$, wheref. the points A, B, O, Q, lie in the circumference of the fame circle; and per Euc. 20, 21, 22, III, $\angle$ BQA = FOA = 2BDA (2IDA), and $\angle$ AQE + AQO = ABO + AQO = 2 right angles, whence $\angle$ AQE = ABO: let each of thefe be taken from a right angle and there will remain $\angle$ AQT = BOS = BCA (BCI); and confequently $\angle$ BQT = AQT = BCA = BDA (IDA). Again from what is deduced in the Analyfis and Euc. 36, III, $FM^2 = FR \times FO = FI \times FQ = FI \times (FI + IQ) = FI^2 + FI \times IQ$: and per prop. 27, B. II. Simpfon's Geom. $FM^2 = FI^2 + mI \times IM$, whence, and per Euc. 35, III, $FI \times IQ = mI \times IM = AI \times IC$, wheref. the points A, Q, C, F, lie in the circumference of the fame circle, and per Euc. 21, III, $\angle$ AQI = FCI, whence, by addition, $\angle$ AQT + AQI = BCI + FCI = 2 right angles; theref. FCB is a right line. Again fince $FI \times IQ = AI \times IC = DI \times IB$, the points F, D, Q, B, lie in the circumference of the fame circle, and $\angle$ BQI = IDF, whence, by addition, $\angle$ BQT + BQI = IDA + IDF = 2 right angles; therefore FDA is a right line. Now having demonftrated that FCB and FDA are right lines: draw the lines BR, CR, and by proceeding as before, it may be fhewn that $\angle$ BRM = CRM = IDC, and $\angle$ BRI = IDE, whence, by addition, $\angle$ BRM + BRI = IDC + IDE = 2 right angles; theref. EDC is a right line; and therefore ABCD is the required trapezium.

The following particulars naturally fuggeft themfelves, from what is deduced in the *Analyfis*, namely,

A ... joining the points of contact of two tangents to the circle ... her of the given points being produced will pafs through th... en point.

Th:

The angle EOF muſt be leſs than a right angle.

For, $OT^2 = OQ \times OE = OR \times OF$; but when EOF is a right angle the points Q, R and O, coincide, and therefore the radius OT becomes $= 0$, or the circle degenerates to a point.

Three points being given in any poſition whatever, ſo that right lines joining them may form an acute angled triangle, a circle may be found having its centre in one of them, ſuch that the oppoſite ſides of a trapezium, inſcribed therein, being produced will interſect in the other two, and have one of its diagonals parallel to a right line given by poſition.

If we imagine the ſhadow of the figure to be projected from a lucid point upon a plane inclined to the plane of the figure, the circle will be projected into a conic ſection with an inſcribed trapezium, whoſe oppoſite ſides produced interſect in two points ſo ſituated with reſpect to the conic ſection, that a right line joining the points of contact of two tangents to the curve from either of the points of interſection, will paſs through the other. This may be gathered from what is deduced in the *Analyſis*, Cor. 1. Prop. 20, B. II. *of Emerſon's Nature and Property of curve lines.*

Whence it follows, that if inſtead of a circle, the curve had been a given conic ſection, the method of conſtruction would have been in no reſpect different:—but then the given points muſt have had ſuch a ſituation, that the right line joining the points of contact of two tangents from either of them would paſs through the other.

The same, answered by the propoſer, Mr. Lowry.

Analysis. Fig. 537, Pl. 28. Suppoſe the thing done, ABCD the trapezium required; P, Q, the given points where the oppoſite ſides produced interſect; and S the point through which the diagonal paſſes. Let O be the centre of the given circle, and draw the tangents PE, QF. Join O, E; O, F; O, P; O, Q; and draw PF to meet the circle at H, and OQ at L; and QE to meet OP at K, PF at I, and the circle at G. Join alſo, P, Q; P, G; and Q, H.

Then becauſe the queſtion requires the trapezium to be inſcribed in the circle, it is evident from *Lawſon's* 33rd propoſition, that the poſition of the points P and Q, with reſpect to the circle, muſt be ſuch that $PQ^2 = PE^2 + QF^2$. Now QF^2 is $= QO^2 - OF^2 = QO^2 - OE^2$; theref. $PQ^2 - PE^2 = QO^2 - OE^2$, and conſeq. PO is perpendicular to EQ. In like manner it may be ſhewn that QO is perpendicular to PF; therefore PG is $=$ PE, and QH $=$ QF: Wherefore *Ibid.* Prop. 18. PE or $HPF = PI^2 + EIG = PI^2 +$

HIF, and QF^2 or $GQE = QI^2 + GIF$. Whence it foll[illegible] from Prop. 34, *Ibid*, that if any ſtraight line PAB be drawn me[illegible] ing the circle at A and B, and AIC, PDC be drawn meeting circle at C and D, the points D, I, B, will be in a ſtraight li[illegible] and if any line QDA be drawn meeting the circle in A and and DIB, QCB be drawn meeting the circle in B and C, points A, I, C, will be in a ſtraight line; and becauſe the po[illegible] P and Q are ſo poſited that a ſtraight line will paſs through [illegible] points A, D, Q, and alſo through the points B, C, Q, it ne[illegible] ſarily follows, that if any trapezium whatever whoſe oppoſite ſi[illegible] when produced, meet in P and Q, be inſcribed in the circle, [illegible] diagonals AC, BD, will always interſect each other in t[illegible] point I(*).

Now becauſe I and S are given points, the diagonal DB will b[illegible] given by poſition, and the *Conſtruction* may be as follows. Draw the tangents PE, QF, and let PF, QE be drawn interſecting at I; through I and S draw the diagonal DISB, join PAB, PDC, QDA and QCB, and it is done, as is manifeſt from the Analyſis.

If only one of the points of interſection had been given, and the other had been required to fall in a line *of any order* given by poſition, the conſtruction would have been very little different from the above. For it is obvious, that the ſtraight lines FIP, EIQ, are the *loci* of the points P, Q.

That the diagonals always interſect each other in the given point I, might be proved differently as follows... For, it is manifeſt from Lawſon's Propoſitions already quoted, that if from P, any two ſtraight lines whatever, as PAB, PDC, be drawn meeting the circle in A, B; D, C; the *locus* of the point of interſection o[illegible] AC, BC, will be the ſtraight line EG; and, ſuppoſing, the point P, and the lines PAB, PDC, to be removed, and from Q any two ſtraight lines whatever, as QDA, QCB, be drawn meeting the circle at A, D; CB; the *locus* of the point of interſectio[illegible] interſection of AC, BD, will be the ſtraight line HF. Now, when the points P, Q, are ſo ſituated that a trapezium can be in ſcribed in the circle, it is obvious, that the point of interſection o[illegible] AC, BD, muſt be a point which is common to each of the *loci* EG, FH, and conſequently it muſt be the point I, where they interſect.

It was from this conſideration that I was firſt led to the diſcovery of the beautiful PORISM exhibited in the above Analyſis, (*), and at that time I was not aware that it had been thought of by any one. Since then a literary friend has favoured me with a copy of the *Opera reliqua* of Dr. ROBERT SIMSON, where I find a compleat inveſtigation is given of this Poriſm.

To ſuch of the readers of the Repoſitory as have not had an opportunity of ſeeing that valuable work, an abſtract of the DR.'s inveſti-

investigation of this *Porism* may perhaps be acceptable. Previous to the investigation he gives the two following propositions.

PROP. I. *Being Prop. LIX. De Porismatibus.*

Fig. 538, Pl. 28. If from two points A, B, without a circle, two straight lines AC, BC, be inflected to the circumference, meeting it again in D and E, and the sum of the rectangles CA, AD; CE, BE, be equal to the square of AB joined; and M be the centre of the circle, and MK be drawn perpendicular to AB, then $BA \cdot AK = CA \cdot AD$.

And the contrary. If the sum of the rectangles CA, AD; CB, BE; be $= AB^2$, and in AB there be taken a point K, such, that $BA \cdot AK = CA \cdot AD$, then if KM be drawn to the centre it will be perpendicular to AB.

Let KM meet the circumference in N,O; then $CA \cdot AD = AK^2 + OK \cdot KN$, and in like manner $EB \cdot BC = BK^2 + OK \cdot KN$; therefore $CA \cdot AD + CB \cdot BE$ (that is, by hypothesis AB^2) $= AK^2 + BK^2 + 2OK \cdot KN$; take away $AK^2 + BK^2$, and $2AK \cdot KB = 2OK \cdot KN$; wherefore $AK \cdot KB = OK \cdot KN$; add AK^2 to each, and $BA \cdot AK = OK \cdot KN + AK^2$, that is $= CA \cdot AD$. In the same way it may be shewn that $AB \cdot BK = CB \cdot BE$.

And if the rectangle $CA \cdot AD + CB \cdot BE$ be $= AB^2$, and in AB the point K be taken making $BA \cdot AK = CA \cdot AD$; then if KM be drawn to the centre, it will be perpendicular to AB.

For, if it is not, from the centre draw MR perpendicular to AB; then by the preceding $BA \cdot AR = CA \cdot AD$, that is $= BA \cdot AK$, which is impossible; therefore MK is perpendicular to AB.

PROP. II. *Being Prop. LX. De Porismatibus.*

Fig. 539, Pl. 28. If from two points A, B, of which A is without, and B within the circle, two straight lines AC, BC, be inflected to the circumference, meeting it again in D and E; and the rectangle $CA \cdot AD$ be equal to the rectangle CB BE, together with the square of AB joined; and from the centre MK be drawn perpendicular to AB, then the rectangle $BA \cdot AK = CA \cdot AD$.

And the contrary, if $CA \cdot AD$ be $= CB \cdot BE + AB^2$, and in AB there be taken the point K, such, that $BA \cdot AK = CA \cdot AD$, then if KM be drawn to the centre it will be perpendicular to AB.

Let AB meet the circumference in S, T; and because the rectangle $CA \cdot AD$, that is, $TA \cdot AS$ is $= CB \cdot BE$ or $SB \cdot BT + AB^2$, let SK^2 be added to each, then (6. II) $AK^2 = SB \cdot BT + AB^2 + SK^2$; take away AB^2, then $2AB \cdot BK + BK^2 = SB \cdot BT + SK^2$. Again, take away, BK^2, and $2AB \cdot BK$ is $= 2SB \cdot BT$; therefore

 AB·BK

AB BK = SB·BT. Add AB² to each, and BA·AK = SB + AB² or CB·BE + AB², that is, by hypothesis, = CA·AD.

The second part of the proposition is proved after the same manner as the second part of the preceding.

Now, the PORISM, as enunciated by Dr. SIMSON, is as follows. Let A, B, be two given points, and to the circumference of a circle CDE given by position, let any two lines AC, BC be inflected meeting it again in D and E, the straight line which joins the points D, E, will pass through a given point.

CASE I. When the straight line which joins the given points A, B, does not meet the circle. Fig. 538.

Let F be the point sought, and let BD be drawn meeting the circle again in G; then because from the points A, B, to the point D, two straight lines AD, BD, are inflected, and which meet the circumference again in C and G; the straight line which joins CG, will, by hypothesis, pass through the same point. Again, draw AG, and because from the points A, B, to G, the straight lines AG, BG, are inflected one of which, as BG, meets the circumference again in D, the straight line which joins the point D, and the point of intersection of the remaining line AG and the circle, will also, by hypothesis, pass through the point F; therefore DF, when drawn, will pass through the point where AG intersects the circle: but DF meets the circumference in E. Wherefore the point E is the intersection of AG and the circumference, that is the points A, G, E, are in a straight line.

Let EH, drawn parallel to AB, meet the circumference again in H, and let DH joined meet AB in K. Then because from the points A, B, the straight lines AC, BC, are inflected to the circumference meeting it again in D and E, and EH is parallel to AB, and DH meets AB at K, the rectangle BA·AK = CA·AD; therefore B, K, D, C are in a circle, and consequently the angle AKD = DCE = AGD (22. III.). therefore the points A, K, G, D, are in a circle; consequently AB·BK = DB·BG = CB·BE. Therefore because it has been shewn that BA·AK = CA·AD, and AB·BK = CB·BE; CA·AD + CB·BE will be = BA·AK + AB·BK, that is, = AB². Let M be the centre of the circle, and let MK, when joined, meet the circumference in N and O, then MK is perpendicular to AB, and consequently to HF (Prop. I.). Therefore, because HE is perpendicular to the diameter NO, and HD is drawn to the circumference and meets the diameter in K, the straight line ED passes through a given point in the diameter, viz. that which divides it in the same

* *Prop. 55. De Porismatibus. The Demonstration is evident*

ratio as NK to KO*. But DE, by hypothesis, passes through a given point F, therefore F is in the diameter: for if not, there would be two points given in the straight line DE, and consequently it would be given by position; but this is contrary to the hypothesis. Therefore DE passes through a given point, when $AB^2 = CA \cdot AD + CB \cdot BE$, and the point F is found by making NK : KO :: NF : FO.

COMPOSITION. Let there be two given points A, B, and to any point in the circumference of the circle CDE given by position, let AC, BC, be inflected meeting it again in D and E, and let $AB^2 = CA \cdot AD + CB \cdot BE$; then DE, when joined, will pass through a given point F, which is found by drawing MK through the centre M perpendicular to AB, and making NK : KO :: NF : FO.

For, let KD be joined meeting the circle again in H, and let DE meet the diameter in F, and join HE: then because $CA \cdot AD + CB \cdot BE = AB^2$, and MK is drawn perpendicular to AB, $BA \cdot AK = CA \cdot AD$; therefore HE is parallel to AB, and consequently perpendicular to diameter NMO. And because HD meets the diameter in K, and DE meets it in F, NK : KO :: NF : FO: but NK and KO are given, therefore the point F is given through which DE passes. *Q. E. D.*

CASE II. When the straight line which joins the given points A, B, meets the circle. Fig. 539.

It may be shewn as in *Case* 1, that, $BA \cdot AK = CA \cdot AD$, and $AB \cdot BK = CB \cdot BE$; therefore $CA \cdot AD = (BA \cdot AK = AB \cdot BK + AB =) CB \cdot BE + AB^2$. and if MK be drawn to the centre it will be perpendicular to AB (Prop. II.), and consequently to HE; therefore because HE is perpendicular to the diameter NO, and HD is drawn to the circumference and meets the diameter in K, the straight line DE passes through a given point in the diameter produced †; but, by hypothesis, DE passes through a given point F, therefore F is in the diameter produced. For, if it is not, two points will be given in the straight line DE, and consequently it would be given by position, but this is contrary to the hypothesis; therefore DE passes through a given point when $CA \cdot AD = CB \cdot BE + AB^2$, and the point F is found by making NK : KO :: NF : FO.

DR. SIMSON then gives the *Composition*, but as it differs so little from that of the first case we shall omit it.

* *Prop. 51, De Porismatibus, or Prop. 156, Lib. 7. Pap. or Prop. 7, Lawson.*

† *Lawson's 7th Proposition.*

IX. QUESTION 279, *by* Amicus.

Two equal bodies A and B move at the ſame time from a point in the periphery of a given circle the one along the periphery and the other along a chord given by poſition.—They move in ſuch a manner that the ſtraight line connecting them always paſſes through the centre of the circle: it is required to determine the nature of the track deſcribed by the centre of gravity of the two bodies?

X. QUESTION 280, *by Miſs* May.

Required the area of the greateſt rectangle that can be inſcribed in the harmonic curve whoſe baſe is 20 and height 2?

XI. QUESTION 281, *by Mr.* John Dawes, *Birmingham.*

Given the ſum of the diſtances between the centres of the ſun (S), moon (L), and a fixt ſtar (A) = 183° 18′; the ∠ ASL = 60°, and the ∠ ALS = 107° 20′, to find each diſtance.

XII. QUESTION 282, *by Mr.* George Roy.

It is, I believe, pretty generally underſtood that the decimal obtained by extracting any root of a whole number cannot be a repetend. An inveſtigation of this is required.

XIII. QUESTION 283, *by Mr.* John Andrew, *Cark.*

Given the difference between the ſegments of the baſe made by the point of contact with the inſcribed circle and that made by the perpendicular, the difference of the ſides and the line biſecting the vertical angle, to conſtruct the plane triangle.

XIV. QUESTION 284, *by Mr.* W. Paſſman, *Hull.*

I demand the diameter of the greateſt circle that can be inſcribed in the quadrant of a given ellipſis?

XV. QUESTION 285, *by Mr.* Nicholas Bottom.

Given the base, the verticle angle, and aggregate of the sum of the sides and line bisecting the vertical angle, to construct the plane triangle.

XVI. QUESTION 286, *by Mr.* Gregory.

In fig. 534, pl. 28, BC is a balance, at one end of which hangs a scale CD in which sits a man, who is counterpoised by the weight W at the end B; BF and FC being equal. Now if the man pushes a staff against the point P (FP being $\frac{2}{7}$ of FC) with a force of 8 stone in the direction DP making the angle DPC = 60°; and the post GH prevent the scale CD from leaving its perpendicular position; it is required to find what weight must be added to, or taken from, W, so that the equilibrium may be maintained, while the pressure on P in the direction DP is exerted?

XVII. QUESTION 287, *by Mr.* Samuel Thornoby.

The same things being supposed as in question 287; it is required to determine when the tradesman's debt will be the greatest possible?

XVIII. QUESTION 288, *by* Merones Minor.

Given $l = 30$, the length; $b = 24$, the breadth; and $h = 4$, the heights of a groin formed by circular arches; to find the superficies and vacuity by a clear and general investigation?

XIX. QUESTION 289, *by Mr.* Johnson, *Birmingham.*

To find a point such that if perpendiculars be drawn from it to the sides of a given trapezium, the sum and sum of the squares of those perpendiculars may be given magnitudes.

XX. QUESTION 290, *by Mr.* Gregory.

For a certain period during every year there is no real night at Cambridge in latitude 52° 12′ 35″ N: it is required to shew the time in the morning when the sun's azimuth from the north is a maximum, on the 1st day of this period in a place whose latitude north is half the complement of the latitude of Cambridge.

XXI. QUESTION 291, *by Mr.* Louis Hill.

It is required to divide a right angle into three others such that their sines may obtain a given ratio?

XXII. QUESTION 292, *by Mr.* Nicholas Bottom.

If a heavy flexible line or chain, of a given length, fastened together at the ends, be suspended by a tack, at a given point upon the perfectly polished surface of a given right cone in such a manner as to include the vertex thereof: it is required to determine the portion of the convex surface included by the chain.

XXIII. QUESTION 293, *by Mr.* James Cunliffe, *Bolton.*

Required the sum of n terms of the series,

$$1\cdot 2x + 2\cdot 3x^2 + 3\cdot 4x^3 + 4\cdot 5x^4 + 5\cdot 6x^5 + \&c.?$$

XXIV. QUESTION 294, *by Mr.* Lowry, *Birmingham.*

Given the sides of a trapezium to construct it when one of the diagonals divides the other in a given ratio.

XXV. QUESTION 295, *by Mr.* Johnson, *Birmingham.*

Let there be any number of straight lines Aa, Bb, Cc, Dd, &c. given by position and parallel to each other in which are the given points A, B, C, D, &c. It is required to draw a straight line to pass through a given point P, and intersect the parallels at L, M, N, O, &c. so that the sum of the squares of AL, BM, CN, DO, &c. may be equal to a given space?

XXVI. QUESTION 296, *by Mr.* James Cunliffe.

To find the least positive values of x and y in whole numbers, so that $x^2 + xy + y^2$ and $x^2 + y^2$ may be both rational squares?

XXVII. QUESTION 297, *by Mr.* John Lowry.

AB and AC are straight lines given by position and P and Q are given points. It is required to draw a straight line through Q

ɛt AB, AC, in B and C, ſo that if BP, CP be joined, the ꞁe BPC may be of a given magnitude; and to ſhew the ?

XVIII. QUESTION 298, *by Mr.* W. Smith, *Liverpool.*

om a given point P, in AB, one of the ſides, produced, of en triangle ABC, it is required to draw a ſtraight line PDE ing the other two ſides AC, BC, in D and E, ſo that if DF rawn parallel to AB, the ſide in which is the given point, ing BC in F, the triangle DEF may be a maximum?

XXIX. QUESTION 299, *by Mr.* Lowry.

om two given points A and B, ſituated in the circumference circle given in magnitude and poſition, it is required to draw ſtraight lines AEG, BEH to interſect the circle at E and two ſtraight lines XG, YH, given by poſition, in which are given points X, Y, at G and H, ſo that the ſum or difference e ſquares of XG, YH may be of a given magnitude.

XXX. PRIZE QUESTION 300, *by* Yanto.

Q be the point in a triangle from which perpendiculars are n to the ſides of the triangle, ſo that the ſum of their ſquares e leaſt poſſible; twice the area of the triangle is a mean proional between the ſum of the ſquares of the ſides of the tri-, and the ſum of the ſquares of the above-mentioned perpenars. Required a Demonſtration?

ARTICLE VI.

SIX PROPOSITIONS FROM LAWSON.

(To be anſwered in Number XIV.)

PROP. LV.

every right-angled triangle, as the difference between the ypothenuſe and one ſide is to the difference between the ſame and its adjacent ſegment, ſo is the ſame ſide to the ſame ent.

PROP.

PROP. LVI.

If HC be a tangent to a circle meeting the diameter DB produced in H, and from the point of contact C a perpendicular CK to that diameter be drawn, and likewise a line from H cutting the circle in F and G, and the perpendicular CK in I, and F be the nearest point to H; then I say that the square of HF is to the square of the tangent HC as FI to IG.

PROP. LVII.

If one side AC of an equilateral triangle ABC be produced to E so that CE may be equal to AC, and from A a perpendicular to AC raised, and from E a line drawn through the vertex B to meet the perpendicular in D; then I say that BD is equal to the radius of the circle which circumscribes the triangle.

PROP. LVIII.

If BD bisect the vertical angle B of a triangle ABC and meet the base in D, and if with either of the other angular points A or C as centre and the adjacent segment of the base as radius a circle be described to cut BD again in E; then I say that BE is to BD as that segment used as radius is to the other.

PROP. LIX.

If BD bisect the vertical angle B of the triangle ABC, and if on BA or BC from B be put a third proportional to the other side and the bisecting line; then I say the rectangle under that side on which it is put and its remainder when the third proportional is taken from it is equal to the square of the adjacent segment of the base made by the bisecting line, that is, BCE $=$ CD^2, or BAE $=$ AD^2.

PROP. LX.

If an isosceles triangle be inscribed in a semi-circle and one of the equal sides produced, and if from any point E in the diameter a perpendicular thereto be drawn to cut the side, the circle, and the side produced in the points G, H, and F respectively; then I say that EG, EH, and EF are continual proportionals.

ARTICLE VII.

To the Editor of the Repository.

SIR,

IF you should look upon the following attempt to investigate the Theorem, given by Dr. Waring, for finding the sum of any power, of the roots of an equation, as deserving of a place in your valuable Repository, I should be glad if you will insert it; but if you know of any paper upon the same subject which will supercede the use of this, I must beg the favour that it may not be printed. The writer of this article having but a very confined acquaintance with Mathematical Books, there may be several demonstrations of this Rule which he has not seen.

I am, Sir, your humble servant,

A. B.

Of the Sums of the Powers of the Roots of an Equation.

LET $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$,

be the given equation, whose roots are a, b, c, d, &c.

It is well known that $\frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c} + \frac{1}{x-d} + \&c.$ is $=$

$$\frac{nx^{n-1} - (n-1)\,px^{n-2} + (n-2)\,qx^{n-3} - (n-3)\,rx^{n-4} + \&c.}{x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c.} =$$

suppose to $M \div N$; therefore,

$$\frac{1}{x-a} - 1 + \frac{1}{x-b} - 1 + \frac{1}{x-c} - 1 + \frac{1}{x-d} - 1 = \frac{M}{N} - n,$$

the number of factors being n; hence

$$\frac{1-x+a}{x-a} + \frac{1-x+b}{x-b} + \frac{1-x+c}{x-c} + \&c. = \frac{M-nN}{N}$$

Multiply the first equation by $x-1$, then

$$\frac{x-1}{x-a} + \frac{x-1}{b} + \frac{x-1}{x-c} + \&c. = \frac{xM-M}{N},$$

add

add two equations together, and

$$\frac{a}{x-a}+\frac{b}{x-b}+\frac{c}{x-c}+\frac{d}{x-d}+\&c.=\frac{xM-nN}{N}$$

$$\frac{px^{n-1}-2qx^{n-2}+3rx^{n-3}-4sx^{n-4}+\&c.}{x^n-px^{n-1}+qx^{n-2}-rx^{n-3}+sx^{n-4}+\&c.};$$

wherefore, by actual division,

$$\left.\begin{array}{l}\frac{a}{x}+\frac{a^2}{x^2}+\frac{a^3}{x^3}+\&c.\\+\frac{b}{x}+\frac{b^2}{x^2}+\frac{b^3}{x^3}+\&c.\\+\frac{c}{x}+\frac{c^2}{x^2}+\frac{c^3}{x^3}+\&c.\end{array}\right\}=\left\{\begin{array}{l}\frac{p}{x}+\frac{p^2-2q}{x^2}+\frac{p^3-3qp+3r}{x^3}\\+\frac{p^4-4qp^2+4rp+2q^2+4s}{x^4}+\&c.\end{array}\right.$$

The homologous terms, or those where x is raised to the same power must be equal to each other, therefore

$a+b+c+\&c.=p$, $a^2+b^2+c^2+\&c.=p^2-2q$.

$a^3+b^3+c^3+\&c.=p^3-3qp+3r$,

$a^4+b^4+c^4+\&c.=p^4+4qp^2+4rp+2q^2$
$-4s$ &c;

for it is evident that the 1st, 2nd, 3rd, &c. numerators in the quotient, are respectively equal to the sum of the 1st, 2nd, 3rd, &c. powers of the roots of the given equation; therefore if the sum of the mth powers of the roots be required, continue the division till the quotient consists of m terms, and the numerator of the last term is the sum required.

It is evident that the *numerator* of the fraction

$$\frac{px^{n-1}-2qx^{n-2}+3rx^{n-3}-4sx^{n-4}+\&c.}{x^n-px^{n-1}+qx^{n-2}+rx^{n-3}+\&c.}$$, is found by multiplying the terms of the given equation respectively by the terms of the following series, viz. 0, — 1, — 2, — 3, — &c.

Because the co-efficients p, q, r, s, &c. are quantities of one, two, three, &c. dimensions, if therefore the sum of the mth powers of the roots be wanted in any equation of n dimensions (n being greater than m) the numerator of the mth term in the quotient given by reducing, or dividing the following fraction, viz.

px

$$\frac{px^{n-1} - 2qx^{n-2} + 3rx^{n-3} \ldots\ldots \mp mwx^{n-m}}{x^n - px^{n-1} + qx^{n-2} \ldots\ldots \pm wx^{n-m}} =$$

$\dfrac{px^{m-1} - 2qx^{m-2} + 3r^{m-3} \ldots\ldots \mp mw}{x^m - px^{m-1} + qx^{m-2} \ldots\ldots \pm w}$, will be the sum required.

From the above it follows as an evident corollary that because

$$\frac{px^{n-1} - 2qx^{n-2} + 3rx^{n-3} - 4sx^{n-4} + \&c.}{x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c.} = \left.\begin{array}{l} \frac{a}{x} + \frac{a^2}{x^2} + \frac{a^3}{x^3} + \frac{a^4}{x^4} + \&c. \\ + \frac{b}{x} + \frac{b^2}{x^2} + \frac{b^3}{x^3} + \frac{b^4}{x^4} + \&c. \\ + \frac{c}{x} + \frac{c^2}{x^2} + \frac{c^3}{x^3} + \frac{c^4}{x^4} + \&c. \\ \&c. \quad \&c. \quad \&c. \quad \&c. \end{array}\right\} \text{in infinitum, therefore, if } x = 1,$$

$$\frac{p - 2q + 3r - 4s + \&c.}{1 - p + q - r + s - \&c.} = \left\{\begin{array}{l} a + b + c + \&c. \\ + a^2 + b^2 + c^2 + \&c. \\ + a^3 + b^3 + c^3 + \&c. \\ \&c. \text{ in infinitum} \end{array}\right.$$

that is, the sum, of the sum of the roots, the sum of the squares, cubes, &c. in infinitum of an equation of n dimensions is equal to $\dfrac{p - 2q + 3r - 4s + \&c.}{1 - p + q - r + s - \&c.}$, if that sum be finite.

To find the sum of the mth powers of their roots:

Put $R = \frac{p}{x} - \frac{q}{x^2} + \frac{r}{x^3} - \frac{s}{x^4} + \&c.$ then because

$\frac{1}{1-R} = 1 + R^2 + R^3 + R^4 + \&c.$ therefore

$$\frac{px^{n-1} - 2q^{xn-2} + 3rx^{n-3} - 4sx^{n-4} + \&c.}{x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4}\ \&c.} =$$

$$\left(\frac{p}{x} - \frac{2q}{x^2} + \frac{3r}{x^3} - \&c.\right) \div \left(1 - \frac{p}{x} + \frac{q}{x^2} - \frac{r}{x^3} + \&c.\right) =$$

$$\left(\frac{p}{x} - \frac{2q}{x^2} + \frac{3r}{x^3} - \&c.\right) \times \left(1 + R + R^2 \ldots R^{m-2} + R^{m-1}\ \&c.\right) =$$

$$C = \begin{cases} (m-3)\ rp^{m-4} \\ +\ (m-3)\left(\frac{m-4}{4}\right) q^2p^{m-5} \end{cases}$$

$$C' = \begin{cases} (m-4)\ rp^{m-5} \\ +\ (m-4)\ \left(\frac{m-5}{2}\right)\ q^2p^{m-6} \end{cases}$$

$$C'' = \begin{cases} (m-5)\ rp^{m-6} \\ +\ (m-5)\ \left(\frac{m-6}{2}\right)\ q^2p^{m-7} \end{cases}$$

&c. &c. &c.

$$D = \begin{cases} -(m-4)\ sp^{m-5} \\ -\ (m-4)\ \left(\frac{m-5}{2}\right)\ \ 2qrp^{m-6} \\ -(m-4)\ \left(\frac{m-5}{2}\right)\left(\frac{m-6}{3}\right) qrp^{m-7} \end{cases}$$

$$D' = \begin{cases} -(m-5)\ sp^{m-6} \\ -(m-5)\ \left(\frac{m-6}{2}\right)\ \ 2qrp^{m-6} \\ -\ (m-5)\ \left(\frac{m-6}{2}\right)\left(\frac{m-7}{3}\right) qrp^{m-8} \end{cases}$$

$$D'' = \begin{cases} -(m-6)\ sp^{m-7} \\ -(m-6)\left(\frac{m-7}{2}\right)\ \ 2qrp^{m-8} \\ -(m-6)\left(\frac{m-7}{2}\right)\left(\frac{m-8}{3}\right) qrp^{m-9} \end{cases}$$

&c. &c.

$$E = \begin{cases} (m-5)\ tp^{m-6} \\ +\ (m-5)\left(\frac{m-6}{2}\right)\ \ (2qs+r^2)p^{m} \\ +\ (m-5)\left(\frac{m-6}{2}\right)\left(\frac{m-7}{3}\right)\ \ 3q^2rp^{m-8} \\ +\ (m-5)\left(\frac{m-6}{2}\right)\left(\frac{m-7}{3}\right)\left(\frac{m-8}{4}\right)\ q \end{cases}$$

$$E' = \begin{cases} + (m-6)\, tp^{m-7} \\ + (m-6)\left(\frac{m-7}{2}\right)(2qs+r^2)\, p^{m-8} \\ + (m-6)\left(\frac{m-7}{2}\right)\left(\frac{m-8}{3}\right) 3q^2rp^{m-9} \\ + (m-6)\left(\frac{m-7}{2}\right)\left(\frac{m-8}{3}\right)\left(\frac{m-9}{4}\right) q^4p^{m-10} \end{cases}$$

&c. &c.

Wherefore by substituting in the equation,

$$a^m + b^m + c^m + d^m = \begin{cases} p \times (A + B + C + D + \&c. \\ -2q \times (A' + B' + C' + D' + \&c. \\ +3r \times (A'' + B'' + C'' + D'' + \&c. \\ -4s \times (A''' + B''' + C''' + D''' + \&c\&c. \end{cases}$$

the values of A, B, C, &c. &c. found above, we shall have as follows

$$p \times (A + B + C + \&c.) = \begin{cases} p^m - (m-2)\, qp^{m-2} \\ + (m-3)\, rp^{m-3} \\ \begin{cases} + (m-3)\left(\frac{m-4}{2}\right) q^2p^{m-4} \\ - (m-4)\, sp^{m-4} \end{cases} \\ \begin{cases} + (m-5)\, tp^{m-5} \\ - (m-4)\left(\frac{m-5}{2}\right) 2qrp^{m-5} \end{cases} \\ \begin{cases} - (m-6)\, up^{m-6} \\ + (m-5)\left(\frac{m-6}{2}\right)(2qs+r^2)\, p^{m-6} \\ - (m-4)\left(\frac{m-5}{2}\right)\left(\frac{m-6}{3}\right) q^3p^{m-6} \end{cases} \end{cases}$$

$$-2q \times (A' + B' + C' + \&c.) = \begin{cases} -2qp^{m-2} + (m-3)\, 2q^2p^{m-4} \\ - (m-4)\, 2qrp^{m-5} \\ \begin{cases} - (m-4)\left(\frac{m-5}{2}\right) 2q^3p^{m-6} \\ + (m-5)\, 2qsp^{m-6} \end{cases} \end{cases}$$

$$+3r \times (A'' + B'' + C'' + \&c.) = \left\{ \begin{array}{l} 3rp^{m-3} - (m-4)3qrp^{m-5} \\ + (m-5)3r^2p^{m-6} \end{array} \right\}$$

$$-4s \times (A''' + B''' + C''' + \&c.) = -4sp^{m-4} + (m-5)4qsp^{m-6}$$

$$+5t \times (A'''' + B'''' + C'''' + \&c.) = +5tp^{m-5} \ \&c.$$

$$-6u \times (A^v + B^v + C^v + \&c.) = \qquad 6up^{m-6}$$

Hence by addition,

$$a^m + b^m + c^m + \&c. = \left\{ \begin{array}{l} p^m - mqp^{m-2} + mrp^{m-3} \\ \left. \begin{array}{l} -ms \\ + m\left(\frac{m-3}{2}\right)q^2 \end{array} \right\} p^{m-4} \\ \left. \begin{array}{l} +mt \\ -m(m-4)qr \end{array} \right\} p^{m-5} \\ \left. \begin{array}{l} -mu \\ + m(m-5)q^5 \\ -m\left(\frac{m-4}{2}\right)\left(\frac{m-5}{3}\right) qs \\ + m\left(\frac{m-5}{2}\right) r^2 \end{array} \right\} p^{m-6} \end{array} \right.$$

It is evident that the theorem may be carried to any nu[mber] of terms by the method given above.

ARTICLE VIII.

To the Editor of the Repository.

SIR,

IF the following paper (for the contents of w[hich] I am indebted chiefly to Montucla's Histoire des Mathematiq[ues] and Bailly's Histoire de L'Astronomie Moderne;) be tho[ught] worthy of a place in your Miscellany, the insertion of it will ob[lige]

Sir, your's, &c.

R. March 1st, 1802. L. J.

An attempt towards a Restitution of the lost Treatise of Eratosth[enes] "de Locis ad Medietates".

[E]RATOSTHENES (borne at Cyrene 271 years before CHRI[ST]) [wa]s of those great and extraordinary men, whose minds em

every ſpecies of human knowledge :—He was called the ſurveyor of the univerſe, the Coſmographer ;—the ſecond Plato. His reputation for learning was ſo great, that he was invited from Athens to Alexandria, by Ptolemy Euergetes, and made by him keeper of the Royal Library at Alexandria, in which employment he died, in the 80th year of his age. It was at his requeſt, that the ſame Ptolemy erected thoſe armillas, or circles of braſs, in the portico at Alexandria, with which Hipparchus, and Ptolemy, afterwards, ſo ſucceſsfully obſerved the heavenly bodies. He was the firſt who attempted to meaſure the Earth, and the method which he invented for this purpoſe, would, alone, have rendered his name immortal. He alſo invented the Aſtrolabe, with which he undertook to meaſure the obliquity of the ecliptic; and it is ſingular that the diſtance of the Sun from the Earth, which he eſtimated at 804,000,000 ſtadia, is exactly conformable to the diſtance aſſigned by Caſſini, and De la Caille. The works, that remain, of this excellent poet, grammarian, mathematician, and aſtronomer, were printed in an 8vo. volume, at Oxford, in the year 1672, and again at Amſterdam, in 1703.

In the mathematics, his attention was principally directed to geometry and aſtronomy : He indeed deſerved to be ranked with the celebrated geometricians of antiquity, *Ariſteus*, *Euclid*, and *Apollonius*, who had written on geometrical analyſis :—*Pappus*, in the preface to his 7th book, mentions a treatiſe of his (now loſt,) in two books, entitled, " de Loci de Medietates", the object of which was to promote that analyſis; it is greatly to be lamented, that Pappups has left us no abſtract of this work; he has given a ſketch of the contents of ſeveral geometrical treatiſes, that are now loſt, but he has done little more than announce the title of this.

Moſt of the loſt treatiſes, of the ancient geometricians, have been reſtored by the moderns, but the reſtitution of *this* has never yet been attempted, by any perſon, that I know of: I ſhall however in what follows give a brief ſketch of what appears to me to have formed the two principal propoſitions in this work, leaving the taſk of demonſtrating them, to others.

It muſt be obſerved, in the firſt place, that, according to Pappus, theſe " *Loci ad Medietates*," were *Conic Sections*; and farther, that, with the ancients, *Mediatate*, (or *Mediety*,) was a *general term*, invented to expreſs three Lines, having either an arithmetical, a geometrical, or an *harmonical* relation; but the words *proportion*, (or ratio,) and analogy were reſtricted to *geometrical relation* only.

This being premiſed, I think it is extremely probable, that the two following theorems have been the ſubject of the loſt work of Eratoſthenes;—If I am miſtaken, I hope my readers will at leaſt give me credit for having noticed two properties of the conic ſections

sections which are not mentioned by any author I have read on that subject.

THEOREM I. *Fig.* 542, *Plate* xxix.

Let there be two right lines AB, CD, any how drawn, let the right line G H, be taken as an axis, and draw the line O F making any angle whatever with G H; then, if the point P (or p, or π, &c.) be so situated, that O P, (or $O\,p$, or $O\,\pi$, &c.) shall be with respect to O F, O E, in any continued arithmetical geometrical, or harmonical mediety, whatever, *i. e.*

if OP : OE :: OE : OF, or if OE : Op :: Op : OF, or if OE : OF :: OF : Oπ; be in any arithmetical, geometrical, or harmonical proportion whatever; I say, that the points P (or p, or π, similarly situated,) shall be in a Conic Section.

THEOREM II. *Fig.* 543, *Plate* xxix.

Let three right lines, AB, CD, GH, be drawn; let any axis LGA be taken at pleasure; and draw the right line OFK, making any angle, whatever with CA; then, if the point P be so situated, that OP : OE :: OF : OI, in any continued arithmetical, geometrical, or harmonical mediety, whatever; I say, that all the points P, &c. so situated, shall be in a Conic Section. And it will be the same with the points p, π, ϖ, &c. i. e.

if OE : OP :: OF : OI; or if OE : OF :: Oπ : OI; or if OE : OF :: OI : Oϖ, be in any continued arithmetical, geometrical, or harmonical proportion, whatever; then will the points p, π, and ϖ be in a Conic Section.

These Theorems may be easily demonstrated by means of the modern analysis; for it is evident that the dimensions of the co-ordinates can never exceed the second degree. It is also evident, that, if they were treated after the manner of the ancients, and if the different cases which they admit of, were all separately examined and investigated, they would afford ample matter for two books, one of which might contain the *first*, its cases, corollaries, &c. and the other the *second*.

For a further account of Eratosthenes, and his writings, see *Vol. 1st* of Montucla's Hist. des Math. before mentioned, page 239—280, new edit. Paris 1799. And Bailly's Hist. de l'Astron. Moderne, B. I. Vol. I. And Phil. Trans. 1772.

ARTI-

ARTICLE IX.

To the Editor of the Mathematical Repository.

SIR,

IF the following observations on the Fraction whose numerator and denominator vanish, should be thought worthy of a place in your valuable Repository, by inserting the same you will greatly oblige,

Sir,

Your most obedient humble Servant,

Hull Citadel, Sept. 8, 1802. W. C.

Observations from a Periodical Publication.

"In this difficulty among mathematicians, I shall observe only, that when a fraction appears by assigning a peculiar value to the variable quantity under the form $\frac{0}{0}$, the fraction in that instance ceases to exist, and the supposition that it can have a determinate value, arose merely from inattention to the fraction in this supposed determinate state.

"For example, $\frac{ax - x^2}{ax^2 - x^3}$ appear in the form $\frac{0}{0}$, when x is equal to a; consequently, to use the common Rule, the fraction has in this case a determinate value, which is * $\frac{a\dot{x} - 2x\dot{x}}{2ax\dot{x} - 3x^2\dot{x}}$, or $\frac{a - 2x}{2ax - 3x^2}$, which, when x is equal to a, becomes $\frac{1}{a}$; that is, the fraction has the determinate value of $\frac{1}{a}$, when x is equal to a. Common sense recoils at this determination, when it puts the algebraical terms into numbers, and making a equal to ten feet, and supposing x equal to it, the fraction appears to be $\frac{100 - 100}{1000 - 1000}$, or nothing at all; and on examining the fraction we see that it really is equal to $\frac{x \times (a - x)}{x^2 \times (a - x)}$, or $\frac{x}{x^2}$, or $\frac{1}{a}$, since $a - x$, divided by $a - x$, gives unity. But this supposed quotient

* The common rule for finding the value of a fraction whose numerator and denominator vanish, being to divide the fluxion of the numerator by the fluxion of the denominator.

tient cannot be true, when $a-x$ ceases to be a number, and in that case the fraction is $\frac{1}{a}\times\frac{0}{0}$, or nothing at all, as it evidently ought to be. The fact shews itself at once to the person who would draw the curve corresponding to the equation $y=\frac{ax-x^2}{ax^2-x^3}$, in which it is evident that all the ordinates will be greater than $\frac{1}{a}$; but as when x ceases to exist, no corresponding ordinate can be drawn to the curve, so no ordinate can be drawn when x becomes equal to a; and annihilates the relationship between y and x. The length of y extends without limit as the abscissa is decreased, it diminishes as x encreases, and its limit of decrease is $\frac{1}{a}$." Thus far the quotation.

Observations on the foregoing, by W. C.

In order to examine how far our writer's reasoning in the above paragraph may be consistent with mathematical demonstration, let us begin where he says, that the quotient of $a-x$ divided by $a-x$ cannot be equal to unity when $a-x$ ceases to be a number; now it is well known by every school boy that whilst $a-x$ remains a number, though infinitely small, the quotient of $a-x$ divided by $a-x$ must be equal to unity; and when $a-x$ ceases to be a number it is manifest that $\frac{a-x}{a-x}=\frac{0}{0}=0^1\div 0^1=0^{1-1}=0^0$, this last expression has been pretty well handled at pages 398 and 399 vol 1. of the Math. Repository and there proved equal to 1; but at the same time the writer, on page 399 ibid. seems to hint that $\frac{0}{0}$ and 0^0 are not the same or equal, however my humble opinion is, that, from above it is very evident the two expressions are equal when 0 does actually represent no number at all, and consequently our question now is to find the value of 0^0 or $\frac{0}{0}$?

First for 0^0

Since the log. of $1-z$ is $-z-\frac{1}{2}z^2-\frac{1}{3}z^3-\frac{1}{4}z^4-$ &c. *ad infinitum* we shall have the log of $\overline{1-z}|^{1-z}$

$=1-z$ into $-z-\frac{1}{2}z^2-\frac{1}{3}z^3-\frac{1}{4}z^4-\frac{1}{5}z^5$

$=z^2+\frac{1}{2}z^3+\frac{1}{3}z^4+\frac{1}{4}z^5+$ &c. $-z-\frac{1}{2}z^2-\frac{1}{3}z^3-\frac{1}{4}z^4$ &c, which when z is equal to 1 will be

1 +

$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$ &c. $-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}$ &c. $=0$, whence the log. of $0^0=0$, and therefore 0^0 is equal to 1.

Secondly for $\frac{0}{0}$.—Becauſe the log. of $1-1=0$ is $-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-$&c, the log. of $\frac{0}{0}$ will be $(-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-$&c.) minus $(-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-$&c) $=0$, which is the log. of 1. Hence we ſee that 0^0 and $\frac{0}{0}$ are each equal to unity; and conſequently 0^0 is equal to $\frac{0}{0}$. Much more might be ſaid on this head but I ſhall leave it to abler hands.

Now to draw the curve correſponding to the equation $y=\frac{ax-x^2}{ax^2-x^3}$; from what has been already done, it will manifeſtly appear that y $(=\frac{ax-x^2}{ax^2-x^3})$ will always be $=\frac{1}{x}$, for when $x=0$, then $\frac{ax-x^2}{ax^2-x^3}=\frac{1}{x}\times\frac{x}{x}\times\frac{a-x}{a-x}$ will be $=\frac{1}{0}\times\frac{0}{0}\times\frac{a-0}{a-0}=\frac{1}{0}$, becauſe $\frac{0}{0}=1$; and when $x=a$, then $\frac{ax-x^2}{ax^2-x^3}=\frac{1}{a}\times\frac{a}{a}\times\frac{0}{0}=\frac{1}{a}$, and for any other value of x than theſe two, it is evident that $y=\frac{1}{x}$, whence, making the ordinate (y) every where $=\frac{1}{x}$, a line being drawn through the extremity thereof will repreſent the curve required.—

How our writer ſhould imagine that the limit of y is $\frac{1}{a}$, I muſt leave to the judgment of the worthy and able correſpondents of the Mathematical Repoſitory, and ſincerely wish ſucceſs may attend the labours of every true lover of the Mathematics.

ARTICLE X.

Anſwer to Queſtion 238 propoſed in No. X. Tranſlated from Golovin's Ruſſian Edition of Euler's Theory on Ship-Building. Communicated by Mr. George Sanderſon.

QUESTION.

IF there be three arches or angles δ, η, ϕ, where δ is given, the

the expression $\frac{\sin.(\delta - \eta - \varphi)}{\sin.\varphi} \times \sqrt{\cos.\eta}$ will be the great

possible when the tan. $(\delta - \eta) = \frac{1}{2}\tan.\eta \times \frac{2 - \tan.\eta \times \tan.}{\frac{1}{2} - \tan.\eta \times \tan,}$

the relation between the angles η and φ being expressed by t equation, cot. $\eta = (a^3 \div 2b^3) \times \tan.\varphi^2$.

Solution.

When any expression is a maximum, its logarithm is a ma mum also. The logarithm of

$\frac{\sin.(\delta - \eta - \varphi)}{\sin.\varphi} \times \sqrt{\cos.\eta} = \text{l. } \sin.(\delta - \eta - \varphi) +$

l. cos. η — l. sin φ.

The fluxion of this being made $= 0$ gives

$$\frac{\text{flux.}(\delta - \eta - \varphi) \times \cos.(\delta - \eta - \varphi)}{\sin.(\delta - \eta - \varphi)} - \frac{\dot{\eta} \times \sin.\eta}{2\cos\eta} - \frac{\dot{\varphi} \times \cos.\varphi}{\sin.\varphi} = 0:$$

But, δ being a constant quantity, by the question, the fluxion of $(\delta - \eta - \varphi)$ will be $= -\dot{\eta} - \dot{\varphi}$; therefore

$$\frac{-(\dot{\eta} + \dot{\varphi}) \times \cos.(\delta - \eta - \varphi)}{\sin.(\delta - \eta - \varphi)} - \frac{\dot{\eta} \times \sin.\eta}{2\cos.\eta} - \frac{\dot{\varphi} \times \cos.\varphi}{\sin.\varphi} = 0.$$

But cot. $\eta = \frac{a^3}{2b^3} \times \tan.\varphi^2$, therefore

l. cot. $\eta = \text{l. } \frac{a^3}{2b^3} + 2\text{l. } \tan.\varphi$; taking the fluxions,

$\frac{-\dot{\eta}}{\sin.\eta \times \cos.\eta} = \frac{2\dot{\varphi}}{\sin.\varphi \times \cos.\varphi}$; this being reduced gives,

$\dot{\eta} = -\frac{2\dot{\varphi} \times \sin.\eta \times \cos.\eta}{\sin\varphi \times \cos.\varphi}$: hence

$-(\dot{\eta} + \dot{\varphi}) = \frac{2\dot{\varphi} \times \sin.\eta \times \cos.\eta}{\sin.\varphi \times \cos.\varphi} - \dot{\varphi}$. Let these values of and $-(\dot{\eta} + \dot{\varphi})$ be written in the expression

$\frac{-(\dot{\eta} + \dot{\varphi}) \times \cos.(\delta - \eta - \varphi)}{\sin.(\delta - \eta - \varphi)} - \frac{\dot{\eta} \times \sin.\eta}{\cos.\eta} - \frac{\dot{\varphi} \times \cos.\varphi}{\sin.\varphi} = 0$, & we

$$\left.\begin{array}{l}\dfrac{2\times\dot{\varphi}\times\text{sin.}\,\eta\times\text{cos.}\,\eta\times\text{cos.}(\delta-\eta-\varphi)}{\text{sin.}\,\varphi\times\text{cos.}\,\varphi\times\text{sin.}\,(\delta-\eta-\varphi)}-\dfrac{\dot{\varphi}\times\text{cos.}(\delta-\eta-\varphi)}{\text{sin.}\,(\delta-\eta-\varphi)}\\ \quad+\dfrac{\dot{\varphi}\times\text{sin.}\,\eta^2}{\text{sin.}\,\varphi\times\text{cos.}\,\varphi}-\dfrac{\dot{\varphi}\times\text{cos.}\,\varphi}{\text{sin.}\,\varphi}\end{array}\right\}=0,\text{ or}$$

$$\left.\begin{array}{l}\dfrac{2\,\text{sin.}\,\eta\times\text{cos.}\,\eta\times\text{cos.}\,(\delta-\eta-\varphi)}{\text{sin.}\,\varphi\times\text{cos.}\,\varphi\times\text{sin.}\,(\delta-\eta-\varphi)}-\dfrac{\text{cos.}\,(\delta-\eta-\varphi)}{\text{sin.}\,(\delta-\eta-\varphi)}\\ \quad+\dfrac{\text{sin.}\,\eta^2}{\text{sin.}\,\varphi\times\text{cos.}\,\varphi}-\dfrac{\text{cos.}\,\varphi}{\text{sin.}\,\varphi}\end{array}\right\}=0.$$

Multiply this equation by $\dfrac{\text{sin.}\,(\delta-\eta-\varphi)}{\text{cos.}\,(\delta-\eta-\varphi)}=\text{tan.}\,(\delta-\eta-\varphi)$. Then

$$\frac{2\,\text{sin.}\,\eta\times\text{cos.}\,\eta}{\text{sin.}\,\varphi\times\text{cos.}\,\varphi}-1+\frac{\text{sin.}\,\eta^2\times\text{tan.}(\delta-\eta-\varphi)}{\text{sin.}\,\varphi\times\text{cos.}\,\varphi}-\frac{\text{cos.}\,\varphi\times\text{tan.}(\delta-\eta-\varphi)}{\text{sin.}\,\varphi}=0$$

Make $P=\dfrac{2\,\text{sin.}\,\eta\times\text{cos.}\,\eta}{\text{sin.}\,\varphi\times\text{cos.}\,\varphi}-1$ and $Q=\dfrac{\text{sin.}\,\eta^2}{\text{sin.}\,\varphi\times\text{cos.}\,\varphi}-\dfrac{\text{cos.}\,\varphi}{\text{sin.}\,\varphi}$.

Then $P+Q\times\text{tan.}\,(\delta-\eta-\varphi)=0$.

But by trigonometry $\text{tan.}\,(\delta-\eta-\varphi)=\dfrac{\text{tan.}\,(\delta-\eta)-\text{tan.}\,\varphi}{1+\text{tan.}(\delta-\eta)\times\text{tan.}\,\varphi}$.

Hence $P+Q\times\dfrac{\text{tan.}\,(\delta-\eta)-\text{tan.}\,\varphi}{1+\text{tan.}\,(\delta-\eta)\times\text{tan.}\,\varphi}=0$, or

$$P+P\times\text{tan.}\,(\delta-\eta)\times\text{tan.}\,\varphi+Q\times\text{tan.}\,(\delta-\eta)-Q\times\text{tan}\,\varphi=0.$$

Hence $\text{tan.}\,(\delta-\eta)=\dfrac{Q\times\text{tan.}\,\varphi-P}{P\times\text{tan.}\,\varphi+Q}$. Let the values of P and Q be reſtored and we ſhall have

$$Q\times\text{tan.}\,\varphi-P=\frac{\text{sin.}\,\eta^2}{\text{cos.}\,\varphi^2}-\frac{2\,\text{sin.}\,\eta\times\text{cos.}\,\eta}{\text{sin.}\,\varphi\times\text{cos.}\,\varphi}$$

$$=\frac{\text{sin.}\,\eta\times\text{cos.}\,\eta}{\text{sin.}\,\varphi\times\text{cos.}\,\varphi}\times(2-\text{tan.}\,\eta\times\text{tan.}\,\varphi),\text{ and}$$

$$P\times\text{tan.}\,\varphi+Q=\frac{2\,\text{sin.}\,\eta\times\text{cos.}\,\eta}{\text{cos.}\,\varphi^2}+\frac{\text{sin.}\,\eta^2}{\text{sin.}\,\varphi\times\text{cos.}\,\varphi}-\frac{1}{\text{sin.}\,\varphi\times\text{cos.}\,\varphi}.$$

But $\text{sin.}\,\eta^2=1-\text{cos.}\,\eta^2$; Therefore

$$P\times\text{tan.}\,\varphi+Q=\frac{2\,\text{sin.}\,\eta\times\text{cos.}\,\eta}{\text{cos.}\,\varphi^2}-\frac{\text{cos.}\,\eta^2}{\text{sin.}\,\varphi\times\text{cos.}\,\varphi}$$

$$=\frac{2\,\text{cos.}\,\eta^2}{\text{sin.}\,\varphi\times\text{cos.}\,\varphi}\times(\tfrac{1}{2}-\text{tan.}\,\eta\times\text{tan.}\,\varphi).$$

Hence $\tan.(\delta - \eta) = \dfrac{Q \times \tan.\phi - P}{P \times \tan.\phi + Q}$

$= \frac{1}{2}\tan.\eta \times \dfrac{2 - \tan.\eta \times \tan.\phi}{\frac{1}{2} - \tan.\eta \times \tan.\phi}$. *Q.*

ARTICLE XI.

Demonſtrations to Lawſon's Propoſitions propoſed in ARTI XXXVII and XLV, VOL. II.

PROP. XXXVII. Fig. 544, 545, *Pl.* 29.

Demonſtrated by Mr. John Lowry.

DRAW FH ⊥ to BC, and join FC. Then by Eu. 13, II,
$AF^2 = CF^2 + AC^2 + 2AC \cdot CH$, but
$CF^2 = AC \cdot CB$, by the hypothеſis; therefore,
$AF^2 = AC \cdot CB + AC^2 + 2AC \cdot CH$; but
$CB + AC = 2DC$; therefore
$AF^2 = 2AC \cdot DH = 2AC \cdot EF$. *Q.*

PROP. XXXVIII. Fig. 546, *Pl.* 29.

Demonſtrated by Mr. John Lowry.

Let ABCD be any regular figure circumſcribed about a whoſe centre is O. Let P be any point within the figur draw the perpendiculars PG, PH, PI, PK, &c. aud joi BP, CP, DP, &c. Now it is evident that the ſum of triangles APB, BPC, CPD, APD, &c. or the rectangle one of the ſides and half the ſum of the perpendiculars PG, PI, PK, &c. is equal to the area of the regular figure A that is, equal to the multiple of the rectangle under one ſi and the ſemi-diameter of the circle by the number of th of the figure. Therefore the ſum of the perpendiculars w equal to the multiple of the ſemi-diameter of the circle number of the ſides of the figure.

PROP. XXXIX. Fig. 547, *Pl.* 29.

Demonſtrated by Mr. Lowry.

Let the ſtraight lines AB, AC, AD, AE, &c. me

rcle in the points B, C, D, E, &c, and make the angles BAC, AD, DAE, &c. equal. Then the arcs BC, CD, DE, &c. equal. Join BD, and ED; then the angle BDE is equal to angle BAF, that is, equal to the angle BAC. Therefore arc BE is equal to the arc BC, and consequently the le is divided into equal parts at the points B, C, D, E, &c.

PROP. XL. Fig. 548, 549, *Pl.* 29.

Demonstrated by Messrs. Cunliffe *and* Lowry.

the triangle DEF draw DG making $\angle$FDG $=$ $\angle$D, and ing EF produced in G. Then $\angle$FDG $=$ $\angle$D, that is by thesis $=$ $\angle$A; also, by Euc. 32, I. $\angle$DFG $=$ $\angle$D$+$ that is by hypothesis, $=$ $\angle$B. Whence it is plain that the BC, DFG. are similar, therefore CA : CB : : GD : GF; e $\angle$GDE is bisected by DF; wherefore, by Euc. 3, VI. GF :: DE : EF, and consequently CA : CB :: DE : EF
Q. E. D.

The same, by Tyro Philomatheticus, *of Hull.*

AB and DF demit the $\perp$s CR and ES; then by sim. $\triangle$s CR :: ED : ES; but the $\angle$CBA being $=$ to both the DE and FED, the $\angle$CBR must be $=$ to $\angle$DFE (Simp. B. I. Prop. 1 and 9); therefore by sim. $\triangle$s CR : CB :: EF, hence CA : CB : : ED : EF. *Q. E. D.*

PROP. XLI. Fig. 550, 551, 552. *Pl.* 29.

Demonstrated by Mr. James Cunliffe, *Fig.* 550.

ACB be a $\triangle$ having the vertical $\angle$C bisected by the line eeting the base in D. Draw AE and BF $\perp$ to CD, and CD in m, and in CD take CF $=$ DF.

evident that the $\triangle$s CAE and CBF, as well as the $\triangle$s and BFD are similar. Therefore

CF :: AE : BF :: DE : DF, and alternately

DE :: CF : DF; therefore by division

$=$ 2mD : DE :: CF — DF $=$ 2mF : DF, or

DE :: mF : DF, and by composition

DE :: mD : DF; consequently

mF :: mE : mD; Whence Euc. 17, VI.

mE $=$ $(m\mathrm{D})^2$, or 2mE $\cdot$ 2mF $=$ 4 $(m\mathrm{D})^2$ $=$ CD^2.

w by the cor. to Prop. 24, B. II. of Emerson's Geom.

$-\mathrm{BD}^2 = 2m\mathrm{F}\cdot\mathrm{CD}$, and $\mathrm{AC}^2 - \mathrm{AD}^2 = 2m\mathrm{E}\cdot\mathrm{CD}$, whence

I 3 (BC²

$BC^2 - BD^2) : (AC^2 - AD^2) = mF \cdot mE : CD^2 = CD \cdot CD$, because $mF \cdot mE = CD^2$, as is already shewn; therefore $BC^2 - BD^2 : CD^2 :: CD^2 : AC^2 - AD^2$. *Q.E.D.*

The same, by Mr. Lowry, Fig. 551.

Let CD be the line bisecting the vertical angle ACB of the triangle ABC, meeting the side AB in D. Make AG, AE, each equal to AD, and BE, BF each equal to DB, and let DG, DH, DE and DF be joined. Then $\angle AGD = ADG = \frac{1}{2}BAC$; therefore the $\angle CDG$ is half the supplement of the $\angle ABF$, that is (because $\angle BDF = DFB$) $= \angle DFC$; therefore the $\triangle$s GDC, DCF are similar. In the same way it is shewn that the $\triangle$s ECD, DCH are similar. Therefore $GC : DC :: DC : CF$, and $EC : DC :: DC : CH$, and consequently, $GC \cdot EC : DC^2 :: DC^2 : CF \cdot CH$; but $GC \cdot EC = AC^2 - AD^2$, and $CF \cdot CH = CB^2 - DB^2$; therefore $AC^2 - AD^2 : DC^2 :: DC^2 : CB^2 - DB^2$. *Q.E.D.*

The same by Tyro Philomathematicus, Hull. Fig. 552.

Make AF = to AD and $\perp$ to CF, and BE = to BD and $\perp$ to CE; then, CD being the line bisecting the vertical $\angle$, CF^2 will be $= AC^2 - AD^2$, and $CE^2 = BC^2 - BD^2$.

Now by Simpson's Geom. B. IV. Prop. 16 and 18, the $\triangle$s AFC and BEC are similar, whence $\angle ECD = \angle FCD$;
again $\angle CED =$ a right-angle (CEB) $+$ BED;
but $\angle BED = BDE = EDC + CDB$,
and the $\angle AFD = ADF =$ a right angle (AFG) $-$ DFG;
therefore the $\angle$ FDG $=$ a right-angle $+$ DFG,
and the $\angle CDF = FDG + CDB$;
Now since the angles ECD and DCF are equal,
the $\angle CED + EDC = CDF + DFG$, that is,
a rt.$\angle$ + CDB + EDC + EDC = rt.$\angle$ + CDB + DFG + DFG,
and taking away the $\angle$s that are common, we shall have
$2\angle EDC = 2DFG$; hence CED = CDF, and the $\triangle$s
CED, and CDF are similar, therefore
$CF : CD :: CD : CE$, or $CF^2 : CD^2 :: CD^2 : CE^2$. *Q.E.D.*

PROP. XLII. Fig. 553, 554, 555. *Pl.* 29.

Demonstrated by Mr. James Cunliffe, *Fig.* 553.

Let FG be a right line intercepted by the tangents, cutting the circle in M and N, and bisected by the chord DE in *m*.

Draw GH parallel to DE meeting the tangent CF in H, and to S,

to S, the centre of the circle, draw the radius DS; alſo draw the reſt of the lines as in the figure.

Becauſe mD and GH are parallel, and $Fm = mG$, it is plain that FD = DH: but SD is ⊥ to DF, by Euc. 18, III; therefore SF = SH = SG; conſequently the triangle FSG is iſoſceles. Now the line Sm, which biſects FG will be perpendicular thereto, as appears from Euc. 12, I; Sn will therefore biſect the chord MN by Euc. 3. III. *Q. E. D.*

The ſame by Mr. John Lowry, *Fig.* 554.

Let AB, AC, be two tangents drawn from the point A and touching the circle in B and C, and let EF be alſo drawn to meet the circle at I and H and the line joining BC at D. Then if FD be = to DE, ID will be = to DH.

Draw FG parallel to BC and let it meet AC produced in G, and join the centre O with the points E, B, F, G, C and D. Then ſince AB = AC, AE is = AG, and conſeq. OF = OG: but DE is = DF (by hypotheſis); therefore by parallels EC = CG, and OC is ⊥ to EG; therefore OE = OG = OF, and conſeq. OD is ⊥ to EF, and therefore DI is = DH.

Converſely. If DI be = DH, FD will be = to DE.

For DO is ⊥ to IH, therefore the points B, F, D, O, are in a circle, and ſo are alſo the points O, D, E, C; therefore the ∠DEO = OCD = OBD = OFD, and OF = OE; therefore ED = DF. *Q. E. D.*

The ſame by Tyro Philomatheticus. *Fig.* 555.

The ∠ AFE being = BFD, and EAF = BFD + BDE,
we have by Prop. XL. FE : EA :: FD : DB,
but FE = FED, therefore EA = DB,
and by Simpſon's Geometry III. 32,
the rectangle GEH = EA^2, and HDG = DB^2;
whence GE = HD, and GF = FH.

Converſely. By Prop. XL, FE : EA :: FD : DB,
$FE^2 : EA^2 :: FD^2 : DB^2$,
but EA^2 = rectangle GEH, and DB^2 = HDG, therefore
EE^2 : GEH :: FD^2 : HDG, and by diviſion
FE^2 — GEH : GEH :: FD^2 — HDG : HDG; that is,
GF^2 : GEH : : HF^2 : HDG, but GF = HF; hence
GEH = HDG, or EA = DB, and conſeq. FE = FD. *Q. E. D.*

PROP.

PROP. XLIII. Fig. 556. *Pl.* 29.

Demonstrated by Messrs. Cunliffe, Lowry, *and* Tyro Philomatheticus.

Let ACD be a △ having its sides AC, DC, equal to each other.

Draw DE and $Cm \perp$ to the sides AC and AD respectively. It appears from Euc. 12. I. that AD is bisected in m. And because DmC and DEC are right angles, the points D, m, E, C are in the circumference of the same circle, therefore by Euc. 36, III. $AC \cdot AE = AD \cdot Am = AD \cdot \frac{1}{2}AD = \frac{1}{2}AD^2$. *Q. E. D.*

PROP. XLIV. Fig. 557, *Pl.* 29.

Demonstrated by Messrs. Cunliffe, Lowry, *and* Tyro Philomatheticus.

First. $$\begin{aligned}(AB+AD)\,BC+BC^2 &= AB\cdot BC + AD\cdot BC + BC^2\\ &= AB\cdot BC + (AD+BC)\cdot BC\\ &= AB\cdot BC + (AB+CD)\cdot BC\\ &= AB\cdot BC + AB\cdot BC + CD\cdot BC\\ &= 2AB\cdot BC + CD\cdot BC.\end{aligned}$$ *Q. E. D.*

Secondly. $$\begin{aligned}(AB+AD)\cdot CD+CD^2 &= (2AD+DB)\cdot CD+CD^2\\ &= 2AD\cdot CD+DB\cdot CD+CD^2\\ &= 2AD\cdot CD+(DB+CD)\cdot CD\\ &= 2AD\cdot CD+BC\ CD.\end{aligned}$$ *Q.E.D.*

PROP. XLV. Fig. 558, 559, *Pl.* 29.

Demonstrated by Mr. John Lowry.

Let ABC be a △, and let CD, CE be drawn to make the ∠s ACD, ECB equal. Through the points D, E, C, describe a circle to meet AC, CB in F and G, and join FG. Then because the ∠s FCD, ECB are equal, the arcs FD, GF are also equal, therefore FG is parallel to AB. Wherefore
AC : BC :: AF : BG, or
$AC^2 : BC^2 :: AC\cdot AF = AD\cdot AE : BC\ BG = BE\cdot BD$. *Q. E. D.*

PROP. XLVI. Fig. 560, 561, 562, 563, 564. *Pl.* 29.

Demonstrated by Mr. James Cunliffe, *Fig.* 560, 561.

First. When the point C is without the circle.

Draw

Draw AD, upon which produced let fall the $\perp$ CF and join BD. The $\angle$ ADB in a semi-circle is a right angle, by Euc. 31, III. wherefore the triangles AED and ADB are similar; therefore $AE : AD :: AD : AB$, whence by Euc. 17, VI.

$AD^2 = AE \cdot AB$. And because CFD and CED are right angles, the points C, E, D, F, lie in the circumference of a circle as appears from Euc. 22, III; therefore by Euc. 36, III.

$AC \cdot AE = AF \cdot AD$: but
$AC \cdot AE = (AB + BC) \cdot AE = AB \cdot AE + BC \cdot AE$
$= AD^2 + BC \cdot AE$, and
$AF \cdot AD = (AD + DF) \cdot AD = AD^2 + AD \cdot DF$;
therefore $AD^2 + BC \cdot AE = AD^2 + AD \cdot DF$, and

by taking away the common square AD^2 from each, there will remain
$BC \cdot AE = AD \cdot DF$: but from what is just deduced,
$AC \cdot AE = AD^2 + AD \cdot DF$, whence by addition
$AC \cdot AE + BC \cdot AE = AD^2 + 2AD \cdot DF$, or
$(AC + BC) \cdot AE = AD^2 + 2AD \cdot DF$; $\therefore$ by Euc. 12, II.
$AC^2 = DC^2 + AD^2 + 2AD \cdot DF = DC^2 + (AC^2 + BC^2) \cdot AE$.

Q. E. D.

Secondly. When the point C is within the circle.
The triangles AED and ADB are similar, therefore
$AE : AD :: AD : AB$; whence
$AD^2 = AB \cdot AE = (AC + BC) \cdot AE = AC \cdot AE + BC \cdot AE$, and
$AD \cdot AF = AC \cdot AE$; therefore
$AD^2 - AD \cdot AF = (AD - AF) \cdot AD$
$= AD \cdot DF = BC \cdot AE$, or
$2AD \cdot DF = 2BC \cdot AE$: but
$AD^2 = AC \cdot AE + BC \cdot AE$, as deduced above. Whence
$AD^2 - 2AD \cdot DF = AC \cdot AE - BC \cdot AE$
$= (AC - BC) \cdot AE$, and theref. Eu. 13, II.
$AC^2 = DC^2 + AD^2 - 2AD \cdot DF = DC^2 + (AC - BC) \cdot AE$.

Q. E. D.

The same by Mr. John Lowry. *Fig.* 562, 563.

$AC^2 = CE^2 + AE^2 + 2CE \cdot EA$; but
$AE^2 + 2CE \cdot AE = (AE + 2CE) \cdot AE = (AC + CB + BE) \cdot AE$ *fig.* 562
or $= (AC - CB + BE) \cdot AE$, *fig* 563.

Also $CD^2 = CE^2 + ED^2$, and $ED^2 = BE \cdot EA$;
Therefore $CD^2 = CE^2 + BE \cdot EA$;
Consequently $AC^2 = CD^2 + (AC \pm CB) \cdot AE$.

Q. E. D.

The

The same by Tyro Philomatheticus. *Fig.* 564.

It is evident that
$AC^2 (=CE^2 + AC^2 - CE^2) = CE^2 + (AC - CE) \cdot (AC + CE)$;
now when C is in the diameter produced
$AC - CE = AE$, and $AC + CE = AC + CB + EB$; whence

$$\begin{aligned} AC^2 &= CE^2 + (AC + CB + EB) \cdot AE \\ &= CE^2 + AE \cdot EB + AE \cdot (AC + CB) \\ &= CD^2 + AE \cdot (AC \cdot CB); \end{aligned}$$

and when C is in the diameter itself,
$AC + CE = AE$, and $AC - CE = AC - CB + EB$; therefore

$$\begin{aligned} AC^2 &= CE^2 + AE \cdot (AC - CB + EB) \\ &= CE^2 + AE \cdot EB + AE \cdot (AC - CB) \\ &= CD^2 + AE \cdot (AC - CB). \end{aligned}$$

Q. E. D.

ARTICLE XII.

Demonstration that the Area of a Parabolic Segment contained under the Abscissa, the ordinate and the curve is equal to two-thirds of the parallelogram contained under the abscissa and ordinate. By WILLIAM FREND, Esq. Fellow of Jesus College, Cambridge.

LET ABC (*Fig.* 578, *Plate* 29.) be a segment of a parabola under the abscissa AB, ordinate BC, and curve AC. Through C draw CD parallel to AB, and through A draw AD parallel to BC. Then the two curvilinear areas ABC, ACD, make up the whole parallelogram ABCD; and if p and q are certain fractional numbers

$$ABC = \frac{ABCD}{p} \text{ and } ACD = \frac{ABCD}{q}.$$

Let the lines AB, AD, be produced to E and I, and through G draw GI parallel to AE and meeting BC produced in H. And join C, G.

Then when the abscissa AB is increased by the magnitude BE and the ordinate BC by the magnitude FG, the area ABC is increased by the mixtilineal area BEGC, and in the same manner the area ACD is increased by the mixtilineal area DCGI. Let the area intercepted between the chord CG and the arc CG be called v; the line $AB = x$, the ordinate $BC = y$, the increment of $AB = m = BE$ and the increment of $BC = n = FG$.

Then

Then the mixtilineal area BCGE $= ym + \frac{mn}{2} + v$,

and the mixtilineal area DCGI $= xn + \frac{mn}{2} - v$.

Let the parameter $= 2a$,

therefore $y^2 = 2ax$ and $x = \frac{y^2}{2a}$,

and $(y+n)^2 = 2a \times (x+m)$;

therefore $y^2 + 2yn + n^2 = 2ax + 2am$,

or $\quad 2yn + n^2 = 2am$

and $\quad \frac{yn}{a} + \frac{n^2}{2a} = m.$

Therefore $\quad ym = \frac{y^2 n}{a} + \frac{yn^2}{2a}$

and $\quad \frac{mn}{2} = \frac{yn^2}{2a} + \frac{n^3}{4a}$;

Therefore $ym + \frac{mn}{2} + v = \frac{y^2 n}{a} + \frac{yn^2}{2a} + \frac{n^3}{4a} + v =$ BCGE

and $\quad xn + \frac{mn}{2} - v = \frac{y^2 n}{2a} + \frac{yn^2}{2a} + \frac{n^3}{4a} - v =$ DCGI.

$$\text{Now } \frac{AEGI}{p} - \frac{ABCD}{p} = \frac{(x+m)(y+n)}{p} - \frac{xy}{p}$$

$$= \frac{(y+n)^2}{2a} \times \frac{y+n}{p} - \frac{y^2}{2a} \times \frac{y}{p}$$

$$= \frac{(y+n)^3}{2ap} - \frac{y^3}{2ap}$$

$$= \frac{3y^2 n}{2ap} + \frac{3yn^2}{2ap} + \frac{n^3}{2ap} = A$$

Make A $=$ BCGE;

Therefore $\frac{y^2 n}{a} + \frac{yn^2}{a} + \frac{n^3}{4a} + v = \frac{3y^2 n}{2ap} + \frac{3yn^2}{2ap} + \frac{n^3}{2ap}$;

now if it is true that the parabolic area AEG is equal to $\frac{AEGI}{p}$, then ſince it will be true whatever is the magnitude of m, the homologous terms on each ſide of the equation will be equal to each other *.

* See Preface to my Principles of Algebra, or the true Theory of Equations eſtablihed on Mathematical Demonſtration. Part the Second, page xii.

Namely

Namely $\frac{y^2n}{a}=\frac{3y^2n}{2ap}$, or $p=\frac{3}{2}$,

$$\frac{yn^2}{a}=\frac{3yn^2}{2ap}, \text{ or } p=\frac{3}{2},$$

$$\frac{n^3}{4a}+v=\frac{n^3}{2ap}=\frac{n^3}{3a};$$

Therefore $v=\frac{n^3}{3a}-\frac{n^3}{4a}=\frac{n^3}{12a}$.

Also $\frac{AEGI}{q}-\frac{ABCD}{q}=\frac{3y^2n}{2aq}+\frac{3yn^2}{2aq}+\frac{n^3}{2aq}=B$.

Make $B = DCGI$.

Therefore $\frac{y^2n}{2a}+\frac{yn^2}{2a}+\frac{n^3}{4a}-v=\frac{3y^2n}{2aq}+\frac{3yn^2}{2aq}+\frac{n^3}{2aq}$.

Now if it is true that the area ACGI is equal to $\frac{AEGI}{q}$, then ſince it will be true whatever is the magnitude of n the homologous terms on each ſide of the equation will be equal to each other*,

Namely $\frac{y^2n}{2a}=\frac{3y^2n}{2aq}$, or $q=3$

$$\frac{yn^2}{2a}=\frac{3yn^2}{2aq}, \text{ or } q=3$$

$$\frac{n^3}{4a}-v=\frac{n^3}{2aq}=\frac{n^3}{6a};$$

Therefore $v=\frac{n^3}{4a}-\frac{n^3}{6a}=\frac{n^3}{12a}$ as before.

Thus the numbers p and q are determined; namely p to be $\frac{3}{2}$ and q to be 3, conſequently the area $ABC=\frac{2}{3}ABCD$, and the area $ACD=\frac{1}{3}ABCD$.

Cor. The area contained between the chord of a parabola and the arc of the parabola may always be found. And if the chord is drawn from the vertex of the parabola this area is equal to the ſquare of the ordinate multiplied into the number, found by dividing the ordinate by twelve times the ſemi-parameter.

* See the preceding Note.

ARTICLE

ARTICLE XIII.

Improved Solutions to some curious Mathematical Problems.

(Continued from Page 10.)

PROB. III. *By Mr.* JAMES CUNLIFFE.

TO find three positive integers, such, that the difference of every two of them may be a square number; and also that the difference between the sum of every two of them and the third may be a square number?

SOLUTION.

Denote the required integers by x, y, and z; and by the question, put $x+y-z=a^2$, $x+z-y=b^2$, and $y+z-x=c^2$.

Half the sum of the first and second of these is $x=\frac{1}{2}(a^2+b^2)$;

Half the sum of the first and third is $y=\frac{1}{2}(a^2+c^2)$;

And half the sum of the second and third is $z=\frac{1}{2}(b^2+c^2)$.

From whence $x-y=\frac{1}{2}(b^2-c^2)$, $x-z=\frac{1}{2}(a^2-c^2)$, and $y-z=\frac{1}{2}(a^2-b^2)$; which three expressions must all be square numbers by the question; in order to effect this, put

$x-y=\frac{1}{2}(b^2-c^2)=e^2$, and $x-z=\frac{1}{2}(a^2-c^2)=d^2$; from whence $y-z=\frac{1}{2}(a^2-b^2)=d^2-e^2=$ a square by the question.

From the last expression $a^2-b^2=2(d^2-e^2)$, which may be divided into factors thus $(a+b)(a-b)=\frac{2s}{r}(d+e)\times\frac{r}{s}(d-e)$.

Put $a+b=\frac{2s}{r}(d+e)$, then will $a-b=\frac{r}{s}(d-e)$;

from which two equations we have

$$a=\frac{s}{r}(d+e)+\frac{r}{2s}(d-e)=\frac{d(2s^2+r^2)+e(2s^2-r^2)}{2rs}, \text{ and}$$

$$b=\frac{s}{r}(d+e)-\frac{r}{2s}(d-e)=\frac{d(2s^2-r^2)+e(2s^2+r^2)}{2rs}.$$

But d^2-e^2 must be a square as observed above, and it is well known that this will be the case when

$d=2rs(m^2+n^2)$, and $e=2rs\times 2mn$, these being substituted for d and e in the above expressions for a and b they become

$a=(m^2+n^2)(2s^2+r^2)+2mn(2s^2-r^2)$, and

$b=(m^2+n^2)(2s^2-r^2)+2mn(2s^2+r^2)$. Now, by means of these and the preceding part of the solution

$$c^2=b^2-2e^2=\left\{(m^2+n^2)(2s^2-r^2)+2mn(2s^2+r^2)\right\}^2-32r^2s^2m^2n^2$$

$$=\left\{\begin{array}{l}(m^2+n^2)^2(2s^2-r^2)^2+4mn(m^2+n^2)(4s^4-r^4)\\ +4m^2n^2(2s^2+r^2)^2-32r^2s^2m^2n^2\end{array}\right.$$

$$=\left\{\begin{array}{l}(m^2+n^2)^2(2s^2-r^2)^2+4mn(m^2+n^2)(4s^4-r^4)\\ +4m^2n^2(2s^2-r^2)^2.\end{array}\right.$$

Dividing by $(2s^2-r^2)^2$ there will be had

$$\frac{c^2}{(2s^2-r^2)^2}=(m^2+n^2)^2+4mn(m^2+n^2)\left(\frac{2s^2+r^2}{2s^2-r^2}\right)+4m^2n^2$$

$$=m^4+4qm^3n+6m^2n^2+4qmn^3+n^4,\text{ by putting } q=\frac{2s^2+r^2}{2s^2-r^2}$$

Assume $m^2+2qmn-n^2=c\div(2s^2-r^2)$, then will

$$\frac{c^2}{(2s^2-r^2)^2}=(m^2+2qmn-n^2)^2$$

$$=m^4+4qm^3n+4q^2m^2n^2-2m^2n^2-4qmn^3+n^4$$

$$=m^4+4qm^3n+6m^2n^2+4qmn^3+n^4.$$

Whence $4q^2m^2n^2-2m^2n^2-4qmn^3=6m^2n^2+4qmn^3$, or $m^2n^2(4q^2-8)=8qmn^3$; Dividing by $4mn^2$

$$m(q^2-2)=n(2q),\text{ or } \frac{m}{n}=\frac{2q}{q^2-2}=\frac{2(4s^4-r^4)}{12s^2r^2-4s^4-r^4}.$$

Whence it appears that m and n may be denoted by any two numbers in the ratio of $2(4s^4-r^4)$ and $12s^2r^2-4s^4-r^4$ respectively, and r and s taken at pleasure.

Now $\frac{c}{2s^2-r^2}=m^2+2qmn-n^2=m^2-n^2+2\left(\frac{2s^2+r^2}{2s^2-r^2}\right)mn$, as assumed above, whence $c=(m^2-n^2)(2s^2-r^2)+2mn(2s^2+r^2)$.

Example. If r and s be taken each $=1$, then m and n will be found to be in the ratio of 6 to 7, or we may put $m=6$, and $n=7$:

then $a=(m^2+n^2)(2s^2+r^2)+2mn(2s^2-r^2)=339$,
$b=(m^2+n^2)(2s^2-r^2)+2mn(2s^2+r^2)=337$, and
$c=(m^2-n^2)(2s^2-r^2)+2mn(2s^2+r^2)=239$.

Whence $x=\frac{1}{2}(a^2+b^2)=114245$, $y=\frac{1}{2}(a^2+c^2)=86021$, and $z=\frac{1}{2}(b^2+c^2)=85345$, which are three numbers that will answer the conditions of the question.

PROB. IV. *By Mr.* James Cunliffe.

ind three square whole numbers, such, that the difference the sum of every two of them, and the third may be a nber.

Solution

Solution.

Let the required ſquares be denoted by x^2, y^2, and z^2, and by the queſtion, put

$x^2+y^2-z^2=a^2$, $x^2+z^2-y^2=b^2$, and $y^2+z^2-x^2=c^2$.

The ſum of the three equations is $x^2+y^2+z^2=a^2+b^2+c^2$, and by taking half the difference between this and each of the former equations there will be had

$x^2=\frac{1}{2}(a^2+b^2)$, $y^2=\frac{1}{2}(a^2+c^2)$, and $z^2=\frac{1}{2}(b^2+c^2)$;

from the two firſt of theſe $a^2=2x^2-b^2=2y^2-c^2$.

Put $x=y+rv$, and $b=c+sv$, then

$$2x^2-b^2=2(y+rv)^2-(c+sv)^2$$

$$=2y^2-c^2+4ryv+2r^2v^2-2scv-s^2v^2=2y^2-c^2.$$

Whence $v=(2sc-4ry)\div(2r^2-s^2)$;

therefore $x=y+rv=\left\{2rsc-y(2r^2+s^2)\right\}\div(2r^2-s^2)$,

and $b=c+sv=\left\{c(2r^2+s^2)-4rsy\right\}\div(2r^2-s^2)$.

Now from what is done $2y^2-c^2=a^2$, and therefore that y, c, and a may be rational, put $(2r^2-s^2)(m^2+n^2)=y$, $(2r^2-s^2)(n^2-m^2+2nm)=c$, and $(2r^2-s^2)(m^2-n^2+2nm)=a$.

The reaſon of theſe laſt aſſumptions will perhaps appear plainer by obſerving that a, y, and c are the roots of three ſquares in arithmetical progreſſion. By means of which, and what is before deduced

$x=2rs(n^2-m^2+2mn)-(2r^2+s^2)(m^2+n^2)$, and

$b=(2r^2+s^2)(n^2-m^2+2mn)-4rs(m^2+n^2)$: therefore

$$z^2=\tfrac{1}{2}(b^2+c^2)=\left\{\begin{array}{l}(4r^4+s^4)(n^2-m^2+2mn)^2+4rs(2r^2+s^2)(m^4-n^4)\\ -8rsmn(2r^2+s^2)(m^2+n^2)+8r^2s^2(m^2+n^2)^2\end{array}\right.$$

$$=\left\{\begin{array}{l}m^4(2r^2+2rs+s^2)^2-4m^3n(4r^4+4r^3s+2rs^3+s^4)\\ +2m^2n^2(4r^4+8r^2s^2+s^4)+4mn^3(4r^4-4r^3s-2rs^3+s^4)\\ +n^4(2r^2-2rs+s^2)^2.\end{array}\right.$$

$$\text{Aſſume}\left\{\begin{array}{l}m^2(2r^2+2rs+s^2)\\ -2mn\left(\dfrac{4r^4+4r^3s+2rs^3+s^4}{2r^2+2rs+s^2}\right)\\ -n^2(2r^2+2rs+s^2)\end{array}\right\}=z,\text{ then will}$$

$$z^2=\left\{\begin{array}{l}m^4(2r^2+2rs+s^2)^2-4m^3n(4r^4+4r^3s+2rs^3+s^4)\\ +2m^2n^2(4r^4+8r^2s^2+s^4)\\ +4mn^3(4r^4-4r^3s-2rs^3+s^4)+n^4(2r^2-2rs+s^2)^2\end{array}\right.$$

=

$$= \begin{cases} m^4 (2r^2+2rs+s^2)^2 - 4m^3 n\,(4r^4+4r^3s+2rs^3+s^4) \\ +\,4m^2 n^2 \left(\dfrac{4r^4+4r^3s+2rs^3+s^4}{2r^2+2rs+s^2}\right)^2 \\ -\,2m^2 n^2 (4r^4+s^4) \\ +\,4mn^3 (4r^4+4r^3s+2rs^3+s^4)\left(\dfrac{2r^2-2rs+s^2}{2r^2+2rs+s^2}\right) \\ +\,n^4 (2r^2-2rs+s^2)^2. \end{cases}$$

This being reduced gives

$$m = n \times \frac{(4r^4+s^4)(4r^4+4r^3s-2rs^3+s^4)-(2r^2+2rs+s^2)(4r^4-4r^3s-2rs^3+s^4)}{(2r^2+s^2)^2(2r^2+2rs+s^2)^2-(4r^4+4r^3s+2rs^3+s^4)^2},$$

which may be farther reduced to

$$m = n \times \frac{4r^3s+4r^2s^2+2rs^3}{4r^4+4r^3s+2r^2s^2+2rs^3+s^4},$$

$$\text{or } \frac{m}{n} = \frac{4r^3s+4r^2s^2+2rs^3}{4r^4+4r^3s+2r^2s^2+2rs^3+s^4}.$$

Hence it appears that m and n may be expounded by any two numbers in the ratio of $4r^3s+4r^2s^2+2rs^3$ and $4r^4+4r^3s+2r^2s^2+2rs^3+s^4$ respectively, and r and s taken at pleasure.

Example 1. Suppose r and s each $= 1$, then $\frac{m}{n} = \frac{10}{13}$, whence we may take $m = 10$ and $n = 13$, by means of which,
$x = 2rs\,(n^2 - m^2 + 2mn) - (2r^2+s^2)(m^2+n^2) = 149$
$y = (2r^2 - s^2)(m^2+n^2) = 269$, and

$$z = m^2(2r^2+2rs+s^2) - 2mn\left(\frac{4r^4+4r^3s+2rs^3+s^4}{2r^2+2rs+s^2}\right) - n^2 (2r^2-2rs+s^2) = -241.$$

or 149, 269, and 241 are the roots of three squares that will answer.

Example 2. If $r = 1$ and $s = -1$, then $\frac{m}{n} = \frac{-2}{1}$, or $m = -2$ and $n = 1$; whence $x = -1$, $y = 5$, and $z = -5$, or the roots of the squares may be 1, 5 and 5, but as two of them are equal they deserve little notice.

I shall now shew how the expressions for x and b might have been obtained by the method of factors.

It was found that $a^2 = \frac{1}{2}(2x^2 - b^2) = \frac{1}{2}(2y^2 - c^2)$, whence $x^2 - y^2 = \frac{1}{2}(b^2 - c^2)$; this expression may be divided into factors thus $(x+y)(x-y) = \frac{r}{2s}(b+c) \times \frac{s}{r}(b-c)$;

put $x+y = \frac{r}{2s}(b+c)$, then will $x-y = \frac{s}{r}(b-c)$; whence

$\frac{r}{4s}(b+c)+\frac{s}{2r}(b-c)=\frac{r^2(b+c)+2s^2(b-c)}{4rs}$

$\frac{b(2s^2+r^2)-c(2s^2-r^2)}{4rs}$, and

$\frac{r}{4s}(b+c)-\frac{s}{2r}(b-c)=\frac{r^2(b+c)-2s^2(b-c)}{4rs}$

$\frac{c(2s^2+r^2)-b(2s^2-r^2)}{4rs}$, and

he last of these $b=\frac{c(2s^2+r^2)-4rsy}{2s^2-r^2}$.

$=\frac{1}{2}(a^2+c^2)$ or $2y^2=a^2+c^2$, that is a^2, y^2, and c^2, are quares in arithmetical progression; therefore we may put

$r^2)(m^2-n^2+2mn)=a$, $(2s^2-r^2)(m^2+n^2)=y$, and

$r^2)(n^2-m^2+2mn)=c$; by means of which

$\frac{2s^2+r^2)-4rsy}{2s^2-r^2}=(2s^2+r^2)(n^2-m^2+2mn)-4rs(m^2+n^2)$, and

$\frac{2s^2+r^2)-c(2s^2-r^2)}{4rs}=2rs(n^2-m^2+2mn)-(2s^2+r^2)(m^2+n^2)$,

re the same expressions as were found the other way.

nswer to the preceding question may be seen in the Gentle-
Diary for 1802, which differs from that here given only in
in the Diary, the values of x, y, and z, are deduced from
a, b, and c, previously obtained, which is not necessary in
ulation.

PROB. V. *By Mr.* JAMES CUNLIFFE.

nd the value of x so that x^2+ax, x^2+bx, and x^2+cx all rational squares.

SOLUTION.

ne $n-x$ for the root of the first; that is, make

$=(n-x)^2=n^2-2nx+x^2$, from whence $=x\ \frac{n^2}{2n+a}$;

g written for x in the second condition gives,

$=\frac{n^4}{(2n+a)^2}+\frac{bn^2}{2n+a}=\frac{n^2}{(2n+a)^2}\times\left\{n^2+b(2n+a)\right\}=$

; therefore $n^2+b(2n+a)=$ a square

 Assume

Assume $s-n$ for the root of this square, and then

$$n^2 + b(2n+a) = (s-n)^2 = s^2 - 2sn + n^2, \text{ whence } n = \frac{s^2-ab}{2(b+s)},$$

$$\text{and } x = \frac{n^2}{2n+a} = \frac{(s^2-ab)^2(b+s)}{4(b+s)^2(s^2+sa)} = \frac{(s^2-ab)^2}{4s(a+s)(b+s)}; \text{ and}$$

this being substituted for x in the third and last condition gives

$$x^2 + cx = \frac{(s^2-ab)^4}{16s^2(a+s)^2(b+s)^2} + \frac{c(s^2-ab)^2}{4s(a+s)(b+s)}$$

$$= \frac{(s^2-ab)^2}{16s^2(a+s)^2(b+s)^2} \times \left\{(s^2-ab)^2 + 4cs(a+s)(b+s)\right\} =$$

a square;

$$\left.\begin{array}{c}\text{therefore}\\ (s^2-ab)^2+4cs(a+s)(b+s)\end{array}\right\} = \left\{\begin{array}{l}a^2b^2+4sabc+2s^2\left\{2c(b+a)-ab\right\}\\ +4cs^3+s^4 = \text{a square.}\end{array}\right.$$

Assume $ab + 2cs - s^2$ for the root of the last square, then will

$$\left.\begin{array}{l}a^2b^2+4sabc+2s^2\left\{2c(b+a)-ab\right\}\\ +4cs^3+s^4\end{array}\right\} \begin{array}{l}= (ab+2cs-s^2)^2\\ = \left\{\begin{array}{l}a^2b^2+4sabc+4c^2s^2\\ -2abs^2-4cs^3+s^4;\end{array}\right.\end{array}$$

Whence $2s^2 \times 2c(b+a) + 4cs^3 = 4c^2s^2 - 4cs^3$, and by transposition $8cs^3 = 4c^2s^2 - 4cs^2(b+a)$; whence $s = \frac{1}{2}\left\{c-(b+a)\right\}$, and by means of this

$$x = \frac{(s^2-ab)^2}{4s(a+s)(b+s)} = \frac{\left\{[c-(a+b)]^2-4ab\right\}^2}{8\left\{c-(a+b)\right\}\times(c+a-b)\times(c+b-a)}.$$

PROB. VI. *By Mr.* James Cunliffe.

To find three square numbers such, that their sum being severally added to their three sides shall make square numbers.

Solution.

In the first place it may be observed that 4, 9, and 36 are three square numbers, whose sum is a square, viz. 49; wherefore let the required squares be denoted by $4x^2$, $9x^2$ and $36x^2$. Now the sum of these being severally added to their three sides gives, $49x^2 + 2x$, $49x^2 + 3x$, and $49x^2 + 6x$, which expressions are all

all to be squares by the question:—or dividing by 49 which is a square number, there will be had

$x^2 + \frac{2x}{49}$, $x^2 + \frac{3x}{49}$, and $x^2 + \frac{6x}{49}$, and these must consequently be all squares.

The question now adapts itself to the general form, in the preceding problem, therefore we may put $a = \frac{2}{49}$, $b = \frac{3}{49}$, and $c = \frac{6}{49}$; whence, and by means of what has been done,.

$$x = \frac{\left\{[c-(a+b)]^2 - 4ab\right\}^2}{8\left\{c-(a+b)\right\} \times (c+a-b) \times (c+b-a)} = \frac{529}{13720};$$

and therefore the roots of the three squares will be

$2x = \frac{1058}{13720}$, $3x = \frac{1587}{13720}$, and $6x = \frac{3174}{13720}$.

*** This is No. 22 of Mr. Bonnycastle's collection of Diophantine Problems.

PROB. VII. *By Mr.* JAMES CUNLIFFE.

To find three square numbers in arithmetical progression, such that if each be added to its respective root, the three sums thence arising shall be rational squares.

SOLUTION.

In the first place $49x^2$, $25x^2$ and x^2 are three squares in arithmetical progression; and each of them being added to its respective root gives

$49x^2+7x$, $25x^2+5x$, and x^2+x, which three expressions are all to be rational squares by the question.

Let the first and second be divided by 49 and 25 respectively, and there will be had

$x^2 + \frac{x}{7}$, $x^2 + \frac{x}{5}$, and $x^2 + x$, which must consequently be all squares.

This question is now adapted to the general form in Prob. V and we may put $a = \frac{1}{7}$, $b = \frac{1}{5}$, and $c = 1$, which gives

$$x = \frac{\left\{[c-(a+b)]^2 - 4ab\right\}^2}{8\left\{c-(a+b)\right\} \times (c+a-b) \times (c+b-a)} = \frac{151321}{7863240};$$

Whence

Whence the roots of the three squares are

$$7x = \frac{1059247}{7863240},\ 5x = \frac{756605}{7863240},\ \text{and } x = \frac{151321}{7863240}.$$

*** The preceding question is answered in the fourth number of the Student.

The following question with various others of a similar nature might be answered by a like method of procedure, if any one would be at the trouble of the calculation, *viz.*

To find three square numbers such that the sum of every two shall be a rational square; and if each be added to its respective root, the sums thence arising shall be rational squares.

PROB. VIII. *By Mr.* JAMES CUNLIFFE.

To divide a given number (n) into four such parts, that the difference of every two of them may be a square number.

SOLUTION.

Let the required parts be denoted by x, y, z, and v,

Then by the question $\left\{\begin{array}{l} x + y + z + v = n \\ x - y = c^2 \\ x - z = b^2 \text{ and} \\ x - v = d^2 \end{array}\right\}$ also $z - y$, $z - v$, and $y - v$ must be squares.

One-fourth of the sum of the four first gives

$x = \frac{1}{4}(n + a^2 + b^2 + c^2)$, and from the second
$y = x - c^2 = \frac{1}{4}(n + a^2 + b^2 - 3c^2)$, and from the third
$z = x - b^2 = \frac{1}{4}(n + a^2 + c^2 - 3b^2)$, and from the fourth
$v = x - a^2 = \frac{1}{4}(n + b^2 + c^2 - 3a^2)$. And by means of these

$$z - y = b^2 - c^2 = \text{a square}$$
$$z - v = a^2 - b^2 = \text{a square, and}$$
$$y - v = d^2 - c^2 = \text{a square.}$$

From whence it appears that the relation of the three squares a^2, b^2, and c^2 must be such, as will make the difference of every two of them a rational square; a^2 being the greatest, and c^2 the least.

Now in one of the solutions to Prob. 1, it was found that a, b, and c may be respectively denoted by 697, 185, and 153, or we may put

$a^2 = 485809$, $b^2 = 34225$, and $c^2 = 23409$; whence, and by means of what is done above,

$x =$

$x = \frac{1}{4}(n + a^2 + b^2 + c^2)$, $y = x - c^2 = \frac{1}{4}(n + 449807)$,
$z = x - b^2 = \frac{1}{4}(n + 406543)$ and $v = x - a^2 = \frac{1}{4}(n - 1399793)$;
where n may be taken at pleaſure.

But the leaſt value of n ſo as to give x, y, z, and v, all poſitive integers will be readily found to be 1399797, and from thence

$x = 485810$, $y = 462401$, $z = 451585$, and $v = 1$.

It is eaſy to perceive, that as (n) may denote any number whatever, we may from what is done, find four numbers ſuch that the difference of every two of them may be a ſquare, and the ſum of all four a ſquare, cube, biquadrate, &c.

PROB. IX. *By Mr.* James Cunliffe.

To find four numbers ſuch, that the ſum of every two of them may be a rational ſquare; and alſo that the ſum of all four may be a rational ſquare.

Solution.

Let x, y, z, and v denote the numbers.

Then by the queſtion
$$x + y = a^2$$
$$x + z = b^2$$
$$x + v = c^2$$
$$y + v = d^2$$
$$y + z = \text{ a ſquare, and}$$
$$z + v = \text{ a ſquare.}$$

The ſum of the firſt and third is $2x + y + v = a^2 + c^2$;
Half the diff. between this and the fourth is $x = \frac{1}{2}(a^2 + c^2 - d^2)$;
Whence $y = a^2 - x = \frac{1}{2}(a^2 + d^2 - c^2)$

$$z = b^2 - x = \frac{1}{2}\left\{2b^2 + d^2 - (a^2 + c^2)\right\}$$

and $v = d^2 - y = \frac{1}{2}(d^2 + c^2 - a^2)$.

By means of the preceding expreſſions,
$y + z = b^2 + d^2 - c^2 =$ a ſquare $= e^2$; and
$z + v = b^2 + d^2 - a^2 =$ a ſquare $= f^2$;
Theſe two laſt give $b^2 + d^2 = c^2 + e^2 = a^2 + f^2$.

From the firſt part of the ſolution to the firſt problem it appears that we may put

$$c = \frac{2rsb - (r^2 - s^2)d}{r^2 + s^2}, \text{ and } a = \frac{(p^2 - q^2)b - 2pqd}{p^2 + q^2},$$

where r, s, p, and q may be expounded by any numbers at pleaſure.

The ſum of the ſecond and fourth equations gives
$x + y + z + v = b^2 + d^2 =$ a ſquare by the queſtion,

Therefore

Therefore we may put $2mn = b$, and $m^2 - n^2 = d$

Whence $c = \frac{2rsb - (r^2 - s^2)d}{r^2 + s^2} = \frac{4rsmn - (r^2 - s^2)}{r^2 + s^2}$

and $a = \frac{(p^2 - q^2)b - 2pqd}{p^2 + q^2} = \frac{(p^2 - q^2)2mn - 2p}{p^2 + q^2}$

and m and n may be taken at pleasure.

Example. If $m = 2$, $n = 1$, $r = 3$, $s = 2$, $p = 4$,
then $b = 2mn = 4$, $d = m^2 - n^2 = 3$,

$$c = \frac{4rsmn - (r^2 - s^2)(m^2 - n^2)}{r^2 + s^2} = \frac{33}{13},$$

$$\text{and } a = \frac{(p^2 - q^2)\,2mn - 2pq\,(m^2 - n^2)}{p^2 + q^2} = \frac{36}{17}.$$

And reducing these values of b, d, c and a to a co nominator, and omitting it, we may put $b = 4 \times 1$ 884, $d = 3 \times 13 \times 17 = 663$, $c = 33 \times 17 =$ $a = 36 \times 13 = 468$. Or, $a^2 = 219024$, $b^2 =$ $c^2 = 314721$, and $d^2 = 439569$, and by means of $\frac{1}{2}(a^2 + c^2 - d^2) = 47088$, $y = a^2 - x = 171936$, z 734368, and $v = d^2 - y = 267633$, which are fou that will answer the question.

By attending to what has been deduced in the fore tion, it will appear that any number whatever, which of being divided into two rational squares, may be di four such parts, that the sum of every two of be a rational square. And this property of bein of being divided into two rational squares is a restr seems indispensibly necessary in the number to be specified above.

ARTICLE XIV.

Answers to the Mathematical Questions proposed in ART
VOL. II.

I. QUESTION 241, *answered by Mr.* G. Buffham.

Put $4x$ and x for the diameters of each vessel; c gallons = 1·631944 cubic feet; $a = (8000 \times 528$

42240012 feet; and $p = \cdot5236$. Since the surface of the liquor must be parallel to the surface of the earth, we have $4x^2 \div a$ and $x^2 \div 4a$ for the heights of two spherical segments, the diameters of whose bases are the same as those of the vessels, the contents of which are easily found to be

$$\left(\frac{64}{a^3}x^6 + \frac{48}{a}x^4\right) \times p \text{ and } \left(\frac{1}{64a^3}x^6 + \frac{3}{16a}x^4\right) \times p.$$

Then by the question, we have the following equation

$$\left(\frac{64}{a^3}x^6 + \frac{48}{a}x^4\right) \times p - \left(\frac{1}{64a^3}x^6 + \frac{3}{16a}x^4\right) \times p = c, \text{ or}$$

$$x^6 + \frac{68}{91}a^3x^4 = \frac{64c}{4095p}, \text{ or in numbers}$$

$$x^6 + 1333262260831756x^4 = 3671164938496296232664.$$

From this equation x is found to be equal to $40\cdot7375$, the less diameter, and $4x = 162\cdot95$ the greater diameter. Whence the heights of the spherical segments are $\cdot00015715$ and and $\cdot00001$ nearly, and their contents $1\cdot6386$ and $\cdot006517$ respectively, which being added to $109486\cdot08585$ the content of either vessel give $109487\cdot724456$ and $109486\cdot092367$ cubic feet respectively, the quantity of liquor in the vessel when full.

The same, answered by Mr. John Whitley, *of Attercliffe Academy, near Sheffield.*

Plate 29, Fig. 565. Let PFBGAEP represent the earth, AB, EF the diameters of the vessels parallel to each other, and intersecting a diameter PG in the points m and n; then it is evident that the given difference will be represented by the zone AEFB, which is known to be equal to $(\text{En}^2 + \text{Am}^2 + \frac{\text{nm}^2}{3}) \times$ mn $\times$ p, where $p = 3\cdot14159$ &c; therefore putting 10 ale gallons $= 2820$ cubic inches $= 1\cdot631944$ cubic feet $= a$, PG $= d$, nm $= x$, En $= y$, and Am $= z$, we shall have this equation, *viz.* $(y^2 + z^2 + \frac{x^3}{3}) \times \frac{1}{2}px = a$. Now it is very evident by the nature of the question that x will be an exceeding small part, and $\frac{x^3}{3}$ will be considerably much more so, therefore the rejecting of $x^3 \times \frac{p}{6}$ will very little affect the value of x, which being done our equation becomes

(y^2

$(y^2 + z^2) \times \frac{1}{2}px = a$, or $y^2 + z^2 = \frac{2a}{px}$: but by the quest.

$4 : 1 :: z : y$, or $4^2 : 1^2 :: z^2 : y^2$, and by composition

$17 : 1 :: z^2 + y^2 = \frac{2a}{px} : y^2 = \frac{2a}{17px}$; consequently

$z^2 = \frac{32a}{17px}$. Also by a known property of the circle,

$Pn \times Gn = y^2$, and $Pm \times Gm = z^2$; hence

$Gm - Pm = \sqrt{(d^2 - \frac{128a}{17px})}$, and $Gn - Pn = \sqrt{(d^2 - \frac{8a}{17px})}$,

and the difference of these is $2mn$, or $2x$; that is,

$$\sqrt{(d^2 - \frac{8a}{17px})} - \sqrt{(d^2 - \frac{128a}{17px})} = 2x.$$

Hence $x = \cdot0001473$ and the greater diameter $= 162\cdot94624$ feet and the lesser $40\cdot73656$ feet; hence the contents are easily found.

II. QUESTION 242, *answered by Mr.* Cunliffe.

Plate 29, Fig. 566. Let ABC represent the cone, HEGF one of the hyperbolic sections parallel to the axis CD. Draw LE ∥ to AB and cutting CD in M; then will CM and ME be respectively equal to the principal semi-transverse and semi-conjugate axes of the hyperbola HEG, as appears from the writers on conics.

Now by the question, the double ordinate GH will be equal to the side of a square inscribed in the circle AHBG; and therefore

$ME = DF = GF = AB \div 2\sqrt{2} = DB \div \sqrt{2}$.

Now by reason of the parallels DE, FE;

$DB : ME = DB \div \sqrt{2} :: \sqrt{2} : 1 :: CD : CM = CD \div \sqrt{2}$;

and by art. 129. Vol. I. Simpson's Fluxions, the area of the hyperbolic space HEG bounded by the double ordinate GH is expressed by

$$CD \times FG - CM \times ME \times \text{h. l.} \frac{CD + \frac{CM \times FG}{ME}}{CM} =$$

$$\frac{CD \times DB}{\sqrt{2}} - \frac{CD \times DB}{2} \times \text{h. l.} (\sqrt{2} + 1) =$$

CD

$$\frac{CD \times DB\sqrt{2}}{2} - \frac{CD \times DB}{2} \times \text{h. l.} (\sqrt{2}+1) = CD \times DB \times \cdot 26642.$$

Again, by art. 157. Vol. I. Simpson's Fluxions, the solid content of the ungula HEGB is expressed by

$$HBG \times \frac{CD}{3} - HEG \times \frac{DF}{3} = HBG \times \frac{CD}{3} - HEG \times \frac{DB}{3\sqrt{2}};$$

and four times this being taken from the whole content of the cone leaves

$$\tfrac{2}{3} \times (DB)^2 \times DC + HEG \times \frac{2DB\sqrt{2}}{3} = DB^2 \times DC \times \cdot 91784984$$

for the content of what remains after the four slices are cut off.

In the present case where $DC = 7\tfrac{1}{2}$ feet and $DB = \frac{18}{12}$, the solid content $DB^2 \times DC \times \cdot 91784984 = 8{\cdot}07899$ cubic feet.

The area of the convex surface of the conical ungula HEGB is expressed by $(CB \div DB) \times$ area of the circular segment HBG as appears by art. 166, Vol. I, Simpson's Fluxions: therefore $(4CB \div DB) \times$ area of the segment HBG will express the convex part of the conical surface taken off by the four slices.

Now the whole convex surface of the cone is expressed by $(CB \div DB) \times$ area of the circle AHBG; from whence deducting the part cut off by the four slices, and there will manifestly remain $(CB \div DB) \times$ area of the square inscribed in the circle $AHBG = (CB \div DB) \times 2DB^2 = 2CB \times DB$.

From whence it appears that what remains of the convex surface of the cone, after the four slices are cut off is geometrically quadrable.

Wherefore if to what remains of the convex surface of the cone after the four slices are taken off there is added the area of the square base, $2DB^2$, together with the area of the four hyperbolic spaces, $4 \times HEG$, there will be had $2CB \times DB + 2DB^2 + 4DB \times CD \times \cdot 26642 = 27{\cdot}4245189$ square feet for the whole superfices of what remains after the slices are taken off.

From what has been deduced we may gather the following curious

THEOREM.

If any right cone is cut by planes parallel to the axis so as just to reduce the base to a plane figure bounded by right lines; then the area of the remainder of the convex surface of the cone will be

geometrically quadrable, its area being expressed by (CB ÷ DB) × the area of the said reduced base.

The same, answered by Mr. Bussham, *Clifton, near Newark.*

The content of the whole cone is easily found = 15927·912 inches.

Now the content of the piece mentioned in the question is equal to the content of the whole cone, minus four times the content of the ungula HEBG.

By a property of the circle and square $1 : \frac{1}{2}\sqrt{2} :: 26 : 13\sqrt{2}$ = GH the side of the square or base of the solid = LE; and AL = $\frac{1}{2}$(AB — LE) = 3·8076118: and by sim. Δs, AB = 26 : DC = 90 :: LE = $13\sqrt{2}$: CM = $45\sqrt{2}$; also EF = 90 — $45\sqrt{2}$ = 26·3604.

Since EFB is a right angle, it is evident, from Dr. Hutton's Mensuration, Cor. 6. Prob. XXVI. Sect. 1, Part III, that the transverse and conjugate axes are 127·27927 and 18·38477 respectively, also the area of hyperbolic section GEH = 311·7117 by Prob. VI. Sect. 7, Part III, of the same book, which call A; also denote the area of the circular segment GBH by *a*.

Then by Page 224 of Hutton ut supra,

$$\frac{\frac{1}{3}h}{D-d} \times \left(a \times D - A \times d \times \frac{DB}{CD}\right) = 491\cdot836326,$$ the content of the slice GEBH, which being multiplied by 4 and the product taken from the whole cone, or 15927·912, leaves 13960·5667 inches, or 8·079 feet nearly, the content of the part remaining after the 4 slices are taken off

Again, the superficies of the whole cone = 26 × 3·1416 × 90·934 ÷ 2 = 3713·817297, and by Prob. XXXII. Sect. 1 Part III, of the above mentioned book, the curve surface of one

$$\text{slice} = \sqrt{\left\{(4h + D - d) \div (D - d)\right\}} \times a = 337\cdot3794$$

this multiplied by 4 gives 1349·5176, which taken from the surface of the whole cone leaves 2364·29969; to this adding four times the area of the hyperbolic section GEH (found above) = 1246·84712 and the area of the square base = $(13\sqrt{2})^2$ = 338 we have 3949·1465 inches, or 27·4246 square feet for the superfices of the part remaing after the four slices are taken taken off.

Neat and ingenious solutions were also received from Messrs. Francis, Marrat, *and* Whitley.

III. QUESTION 243, *answered by Miss* Susan May.

Plate 29, Fig. 567. Let ACB be the triangle required; AD, BE, and CF the ⊥s demitted from the angles on the opposite sides, and which meet in a point, as at G. Join DE, DF, and EF. Then because the points B, F, G, D are in a circle, and also the points C, D, G, E, we have ∠GBF = GDF, and ∠ECG = EDG: but ∠GBF = GCE, therefore EDG = FDG; and in like manner it may be shewn that DEG = GEF and GFE = GFD.

Now in the △ DEF the three sides are given, *viz.* EF = 9, ED = 10, and FD = 11; therefore the three ∠s are found by trigonometry, *viz.* DFE = 58° 59′ 42″, EDF = 50° 28′ 50″, and DEF = 70° 31′ 28″; therefore ∠ ADF = 25° 14′ 25″; ∠DFA = 119° 29′ 52″, and DAF = 35° 15′ 45″; hence AD is found = 16·585, BD = 11·724, and CD = 9·386. Consequently $(CD + DB) \times \frac{1}{2}AD = 21·11 \times \frac{16·585}{2} = 175·04$ square chains = 17½ acres, the content required.

The same, answered by Tyro Philomatheticus, *of Hull.*

Plate 29, Fig. 568. Let ABC represent the triangular field, AD, BE, and CF the ⊥s from the ∠s on the opposite sides, G the point of intersection, and ED, EF, FD the given lines. Before I proceed to the calculation it may be thought necessary to prove (as I have not yet seen it done) that, *The perpendiculars from the angles on the opposite sides bisect the angles of the triangle formed by joining the extremities of those perpendiculars.*—On AB describe a semicircle, then because of the right ∠s AEB and ADB, it will pass through the points E and D; and AEG, AFG being right angles it is manifest that the points A, F, G, and E are in the circumference of a circle, and the ∠ EFG = EAG; again BFG, BDG, being right ∠s, it is likewise evident that B, F, G, and D are in the circumference of a circle, and ∠DFG = DBG: but the ∠s EAG and DBG, standing on the same segment ED, are equal; therefore EFG = DFG; and in a similar manner it is easily proved that FDG = EDG, and that DEG = FEG.

Now ED = 9, EF = 10, and FD = 11, whence the ∠s by Trig. are found, *viz.* FED = 70° 32′, EDF = 58° 59½′ and EFD = 50° 28½′ from which, and what is done above, we get ABC (= comp. of ½ FED) = 54° 44′, CAB (= comp.

 of

of $\frac{1}{2}$ FDE) $= 60° 30\frac{1}{4}'$, and BCA ($=$ comp. of $\frac{1}{2}$EFD) $= 64° 45\frac{3}{4}'$; and by trigonometry, as
fine of $\angle$ A : EF :: cos. $\frac{1}{2}$FED : AF $=$ 9·3808, and
fine of $\angle$ B : FD :: cos. $\frac{1}{2}$EDF : FB $=$ 11·7264;
therefore AB $=$ 21·1069; again as
fine of $\angle$ A : EF :: cos. $\frac{1}{2}$EFD : AE $=$ 10·3925, and
fine of $\angle$ C : ED :: cos. $\frac{1}{2}$EDF : EC $=$ 8·66015;
therefore AC $=$ 19·05265; hence the area $=$
AB $\times$ AC $\times$ $\frac{1}{2}$ fine $\angle$A $=$ 175·01 chains nearly, or $17\frac{1}{2}$ acres.

The fame, anfwered by Mr. John Whitley.

Plate 29, Fig. 569. Let ABC reprefent the piece of ground: from the $\angle$s, to the oppofite fides, draw the $\perp$s AE, BF, CD; draw alfo the lines DF, DE, EF, interfecting the $\perp$s in n, p, m, refpectively. Since AEB and AFB are right $\angle$s, the points A, F, E, B are in a circle, and for the fame reafon the points C, E, D, A are in a circle. Therefore $\angle$ ABF $=$ AEF, and ACD$=$AEB: but $\angle$ABF$=$ACD, becaufe each of thefe is the complement of the fame $\angle$A; therefore $\angle$AEF $=$ AED, and confequently the $\perp$ AE bifects the $\angle$FED. In the fame manner it may be proved that the $\angle$s FDE, and DFE are refpectively bifected by the $\perp$s CD and BF.

Having premifed this the calculation may be as follows.
Put DE $= 9$, FD $= 11$ and EF $= 10$; then by Euc. 3, VI,
FE $+$ ED : FD :: FE : Fn $= 110 \div 19$, whence
D$n$$=FD-Fn$$=11-110\div19=99\div19$, and by Em. Geo. 26, II,
En^2 $=$ FE$\times$ED$-$F$n\times$D$n$$=21600\div19^2$, or E$n$$=60\sqrt{6}\div19$.
Alfo FE $+$ Fn : En :: FE : EO $= 2\sqrt{6}$, whence
nO $=$ En $-$ EO $= 22\sqrt{6}\div 19$.
Becaufe ALO, AFO are right $\angle$s, the points A, F, O, D are in a circle; therefore, Euc. 35, III,

$$An\times nO = Fn\times nD, \text{ or } An = \frac{Fn\times nD}{nO} = \frac{5\times33\sqrt{6}}{2\times19}, \&c.$$

$$AO = An + nO = \frac{5\times33\sqrt{6}}{2\times19} + \frac{22\sqrt{6}}{19} = \frac{11\sqrt{6}}{2}. \quad \text{Again}$$

DF$+$FE : DE :: DF : Dp $= 33\div7$, whence pE$=30\div7$; &
F$p^2$$=DF\times$FE$-Dp\times pE=440\div49$, or F$p$$=20\sqrt{11}\div7$. Alfo
FE$+$Ep : Fp :: FE : FO$=2\sqrt{11}$, and by the fim. $\triangle$s, FOE, AOB;
FO : FE :: AO : AB $= 5\sqrt{66}\div 2$. Again
FD $+$ DE : FE :: FD : Fm $= 11\div 2$, whence mE $= 9\div 2$.

Wherefore

Wherefore $Dm^2 = FD \times DE - Fm \times mE = \frac{3}{4} \times \frac{99}{1} = \frac{9}{4} \times 33$,

or $Dm = \frac{3}{2}\sqrt{33}$. Also

$DE + Em : Dm :: DE : DO = \sqrt{33}$; whence $Om = \frac{1}{2}\sqrt{33}$.

And $Cm \times mO = Fm \times mE$, or $Cm = \frac{Fm \times mE}{mO} = \frac{3}{2}\sqrt{33}$.

Therefore $Cm = Dm$ and $DC = 3\sqrt{33}$. Whence the area is $\frac{AB \times DC}{2} = \frac{5}{2}\sqrt{33} \times \frac{3}{2}\sqrt{66} = \frac{15}{4} \times 33\sqrt{2} = \frac{15}{4} \times 33 \times 1{\cdot}41421356 = 175{\cdot}008928$ square chains $= 17{\cdot}5$ acres, nearly.

And thus the queſtion was anſwered by Mr. Cunliffe;

Who obſerves "What is deduced previous to the calculation ſuggeſts an eaſy way of conſtructing the problem geometrically, and which may be as follows, *viz.*—With the given diſtances of the points, where the perpendiculars from the angles meet the oppoſite ſides, let the triangle FDE be formed, and having biſected the angles FDE, FED, and DFE with the lines *Dm*, *En*, and *Fp*, draw the lines AB, BC, and AC reſpectively perpendicular to the former, through the points D, E, and F, and they will form the triangle required".

Solutions were alſo received from Meſſrs. Francis, Lowry, *and* Swale.

IV. QUESTION 244, *anſwered by Mr.* Cunliffe.

Plate 29, Fig. 570. Let the triangle FAE repreſent a vertical ſection of the wall at right angles to the face next to the water, and let the point S in AE denote the ſurface of the water. Draw FP to any point P*m* in the line AS. By the principles of Hydroſtatics, the preſſure of the water at P againſt AE will be expreſſed by $m \times SP$, where m denotes the ſpecific gravity of the water; and by the reſolution of Forces, its force in a direction perpendicular to FP will be expreſſed by $m \times SP \times (PA \div FP)$: and therefore the effect of the preſſure of the water againſt the point P to turn the triangle FAE about F will be expreſſed by
$m \times SP \times (PA \div FP) \times FP = m \times SP \times PA$.

Conſequently the effect of the preſſure of the water againſt the whole line AS, to turn the triangle FAE about F will be expreſſed by the ſum of all the $m \times SP \times PA$.

 Now

Now to find the sum of all the $m \times SP \times PA$, put $AS = d$, $SP = x$, then $PA = d - x$; and the sum of all the $m \times SP \times PA =$ flu. $m x\dot{x} (d - x)$ when $x = d$: but flu. $m x\dot{x} (d - x) =$ flu. $m d x\dot{x}$ — flu. $m x^2 \dot{x}$

$$= m \times \left(\frac{dx^2}{2} - \frac{x^3}{3}\right),$$

and when $x = d$, the fluent becomes

$m \times \left(\frac{d^3}{2} - \frac{d^3}{3}\right) = \frac{md^3}{6} = \frac{m}{6} \times AS^3$, which as is shewn above expresses the effect of the pressure of the water upon the line AS to turn the triangle FAE about F.

To find the effect of the weight of the triangle FAE to turn it in a contrary direction. From G, the centre of gravity of the triangle, draw GN perpendicular to FA, and let W denote the weight of the triangle. Then by the principles of mechanics, the effect of the weight of the triangle FAE to turn it about F in a direction contrary to that of the action of the water will be expressed by

$$\begin{aligned} W \times FN &= \tfrac{1}{2} \times FA \times AE \times n \times FN \\ &= \tfrac{1}{2}n \times FA \times AE \times FN \\ &= \tfrac{1}{3}n \times FA^2 \times AE; \end{aligned}$$

because $W = \frac{1}{2}n \times FA \times AE$ and $FN = \frac{2}{3} \times FA$; where n denotes the specific gravity of the wall.

Therefore when the wall just supports the fluid, the two preceding forces must be equal to each other, *viz.*

$\frac{1}{3}n \times FA^2 \times AE = \frac{1}{6}m \times AS^3$ whence

$FA^2 = AS^2 \times \frac{m \times AS}{2n \times AE}$, or $FA = AS\sqrt{\frac{m \times AS}{2n \times AE}}$.

In the present case where $AS = 10$, $AE = 12$, $m = 7$, and $n = 11$,

$FA = AS\sqrt{\frac{m \times AS}{2n \times AE}} = 10\sqrt{\frac{35}{132}} = 5\cdot149286 = 5\frac{3}{20}$

feet nearly, the thickness of the wall at the bottom, just sufficient to support the fluid.

Messrs. Buffham *and* Marrat *also sent answers to this question.*

V. QUESTION 245, *answered by Mr.* Gregory, *the Proposer.*

Plate 29, Fig. 571. It has been demonstrated that the altitude AK of the greatest inscribed triangle is equal to $\frac{3}{4}$ of the transverse diameter; it has also been demonstrated that the altitude LB of the least circumscribing triangle is $\frac{3}{2}$ of the transverse: And

it will readily appear, since LF is half of LB, that EF = HK is = $\frac{1}{4}$BM. Therefore, the triangles LMN and AHI are similar, and as the altitude of the former is double that of the latter, its area is quadruple; and the difference between the areas is equal to thrice AHI. Hence 70·148025 the given difference divided by 3 gives 23·382675 area of AHI. Let this be divided by 9 = AK = $\frac{1}{4}$AB, the quotient is 2·598075 = $\frac{3}{2}\sqrt{3}$ = HK.

Then, by the nature of the ellipse,

$AK \times KB : HK^2 :: AG \times GB : GC^2$;

that is as $27 : \frac{27}{4} :: 36 : \frac{36 \times 27}{27 \times 4} = 9 = GC^2$.

Consequently GC = 3 and CD = 6 the conjugate required.

The same, answered by Miss Susan May.

Plate 29, Fig. 572. Let BEAF represent the ellipsis required, QPR the least circumscribing triangle, and EBF the greatest inscribed one.

Now it is manifest from the first solution to quest. 170 of Repository, that

GP = GB = $\frac{3}{4}$BA = 9, and

$\triangle$QPR=27×GE (BR×BP)=4$\triangle$EFB (GE×GB); therefore 3GE × GB=70·148025, or GE=70·148025÷27=2·598075.

And by the property of the ellipse,

$(\text{conjugate})^2 : 12^2 :: GE^2 : BG \times GA = \frac{3}{16}(12)^2$, that is,

the conjugate $= 4GE \div \sqrt{3} = 10{\cdot}3923 \div 1{\cdot}73205 = 6$.

The same, answered by Mr. William Marrat, *Boston.*

Plate 29. Fig. 572. Let AFB be half the eliptical table, and draw the lines as per figure. The greatest inscribed $\triangle$BFG is when the altitude BG is $\frac{3}{4}$ of the transverse axe AB; and the least circumscribing $\triangle$ (by Prob. 5, Emerson on Curve Lines) is, when the subtangent PG = GB; and by Prop. IX, Ellipse, Hutton's Conics,

PB : PO :: PG : PA :

and alternately PB : PG :: PO : PA;

but PB = 2PG, therefore PO = 2PA, and conseq. AG = GO.

Now GB = $\frac{3}{4}$AB = 9, and PB = 18, therefore putting GF = x, twice the area of the $\triangle$ BFG = $9x$, and twice that of the $\triangle$ PBR = $36x$; therefore by the question $36x - 9x =$ 70·148025; hence $x =$ 2.598075.

Then, by the property of the ellipse,
$AG \times GB : GF^2 :: AB^2 : CD^2 = 36$.
Consequently $CD = 6$, the conjugate diameter required.

The same, answered by Mr. G. Buffham.

Plate 29, Fig 572. Let the ellipse AEBF represent the table, O its centre, and let lines be drawn as in the figure. Put t and c for the transverse and conjugate diameters respectively; put also $BG = x$. Then, by conics, $GE = (c \div t) \sqrt{(tx - x^2)}$, and the area of the $\triangle$ $BEF = (c \div t) x \sqrt{(tx - x^2)}$, which by the question must be a maximum, or $tx^3 - x^4$ a maximum, which being fluxed and reduced gives $x = \frac{3}{4}t = 9$; and, by substitution, $(c \div t) x\sqrt{(tx - x^2)}$, becomes $\frac{3}{16}tc \sqrt{3}$.—By the Scholium to Prob. 8. Simpson on the maxima et minima of geometrical quantities, tangents drawn to the ellipse through the points E and F form the least circumscribing $\triangle$; for, by the said scholium, it is a minimum when the subtangent $GP = BG$; and per conics,
$OG : GA :: BG : GP$; therefore $OG = GA$, or
$BG\ (x) = \frac{3}{4}AB\ (\frac{3}{4}t)$ as before

Whence the area of the triangle PQR is easily found to be $(4c \div t)\, x\sqrt{(tx - x^2)} = 3c\sqrt{27}$, by substituting for x and t. Then per question $3c\sqrt{27} - (3 \div 16)\, tc\sqrt{3} = 70{\cdot}148025$. Hence $c = 6$ the conjugate required.

The same again, by Mr. John Whitley.

Plate 29, Fig. 572. CONSTRUCTION. Let ABCD represent the elliptic table, AB the given transverse, and CD the required conjugate diameter, and SLMN a rectangle equal in area to the third part of the given difference of the triangles. Upon AB, as a diameter, describe the circle AIBH, and bisect its radius AO in G, and draw IGH $\perp$ to AB, meeting the circle in I and H; in IG take the point F so that $BG : MN :: ML : GF$, make OC ($= OD$) perpendicular to AB and such that $IG : FG :: AO : OD$, so shall CD be the required conjugate

DEMONSTRATION. Join IB, HB, and in BG, FG, produced take PG and GE respectively equal to BG and FG. Join also PF, PE, which produce to meet RBQ drawn parallel to FE in Q and R, also join BF, BE. Then by *construction* $AG = GO$, therefore the $\triangle$ BHI is equilateral, and is, therefore, the greatest that can be inscribed in the circle; and because $PG = GB$, PE

PE = EQ = BE, therefore the △s PFE, BFE, BEQ, and BFR are equal in all refpects. But by the fcholium to Theorem 8, Simpfon's Geometry, on the max et min. of quantities, the △ BFE is half of the greateft parallelogram that can be infcribed in the triangle PQR; it alfo follows from the faid fcholium that the triangle PBR is the leaft that can be circumfcribed about the curve ACB, therefore the △ BFE which is double of the △ BFG is a maximum, alfo the △ PQR which is double the △ PBR is a minimum. And by a property of the ellipfe,

$AG \times GB = IG^2 : FG^2 :: AB^2 : CD^2$, wherefore

$IG : FG :: AB : CD$; but by conftruction

$BG : NM :: ML : GF$, therefore, Euc. 16, VI,

$BG \times FG = SL \times LM =$ the area of the triangle FBE (per *conftr.*)

$= \frac{1}{4}\,PQR = \frac{1}{3}\,PQR - \frac{1}{3}\,BFE.$ *Q. E. D.*

The fame otherwife, by Mr. Cunliffe, *of Bolton.*

It is well known that an equilateral △ is the greateft that can be infcribed within, and the leaft that can be circumscribed about any given circle. Therefore if a circle with its greateft infcribed, and leaft circumscribing △s, be imagined to be projected *orthographically* upon a plane which makes an ∠ with the plane of the figure, the projection will be an ellipfis with its greateft infcribed and leaft circumfcribing △s. And the difference between the greateft infcribed and leaft circumfcribing △s in the circle will be to the difference between the greateft infcribed and leaft circumfcribing △s in the ellipfis, as the tranfverfe to the conjugate diameter of the ellipfis. This is well known to thofe acquainted with the principles of the orthographic projection.

This being premifed, let d denote the tranfverfe, and c the conjugate diameters of the ellipfis; then it is well known that $\frac{1}{2}d\sqrt{3}$ will exprefs the fide of an equilateral △ infcribed in the circle whofe diameter is d; $(3 \div 16)\,d^2\sqrt{3}$ its area. Alfo $d\sqrt{3}$ will exprefs the fide of the leaft circumfcribing equilateral △, which is manifeftly double the fide of the infcribed △; therefore its area will be four times the area of the infcribed △, and confequently the difference between the areas of the faid △s will be equal to thrice the area of the infcribed △ or $(9 \div 16)\,a^2\sqrt{3}$. Now let A denote the difference of the areas of the greateft infcribed and leaft circumfcribing △s in the ellipfis;

then $d : c :: (9 \div 16)d^2\sqrt{3} : A$; whence $(9 \div 16)d^2c\sqrt{3} = dA$,

or $c = 16A \div 9d\sqrt{3} = (16A \div 27d) \times \sqrt{3} = 6$ feet.

VI. QUESTION 246, *anfwered by Mr.* Bofworth, *the Propofer.*

In Fig. 573, Pl. 29. Let AB and AC reprefent the two mirrors, and a the fmall object on AB, placed 6 inches from B. It

has been proved that the diſtance *bc* of the image formed by reflection from AC, is equal to the diſtance of the object *a* from that mirror; and that the images *l*, *g*, *c*, *e*, *i*, formed by continual reflections from both mirrors are in the circumference of a circle whoſe centre is A and whoſe radius is the diſtance of the object *a* from that centre. See *Enfields Inſtitutes of Nat. Phil.* Prop. CI, and CII; and *Wood's Optics*, Prop. XIII. XIV. From the known principles of optics we are certain that *ac*, *eg*, and *il*, are perpendiculars to AC, and biſected by it; in like manner *ce*, *gi*, and *lm*, are perpendiculars to AB, by which they are biſected. It follows, then that the ∠s *ace*, *ceg*, *egi*, *gil*, *ilm*, are each = the given ∠ CAB = 15°; and, by Euc. III. 20. the arcs *lg*, *gc*, *ca*, *ae*, *ei*, *im*, each = 30°—whence the number of images formed on the two mirrors = 180 ÷ 30 = 6; but the laſt is not perfect, as the centre of it falls on A, the point of interſection, and therefore only half of it is caught by the mirror. If twice the ∠ of inclination be not a meaſure of 180°,—the number of images formed will be that whole number which is next inferior to the quotient of 180° by twice the angle.

To find the diſtance of each image behind the ſurface of the mirror upon which it is formed, we muſt conſider that the firſt diſtance is denoted by the ſine of the ſame number of degrees as the inclination of the mirrors; the ſecond is the ſine of twice that number; the third of three times, and ſo on—for *bc* is, evidently, the ſine of 15°, in the preſent caſe; *de* of 30°—and ſo of the reſt. Theſe diſtances may eaſily be calculated by a table of natural ſines; thus,

			In.	Inches.
As 1 (Rad) :	·2588190 = ſin. of 15°	::	36 (Rad) :	9·3172840 = *bc*
	·5000000 = ...30°	::	36	: 18· .... = *de*
	·7071068 = ...45°	::	36	: 25·4558448 = *fg*
	·8660254 = ...60°	::	36	: 31·1769144 = *hi*
	·9659258 = ...75°	::	36	: 34·7733288 = *kl*
				36· .... = *Am.*

The ſame, anſwered by Miſs Suſan May.

Plate 29, Fig 574, 575. Let CA′, CB′, be two plain mirrors inclined in a given angle A′CB′, and let *a* be an object placed between them. With the diſtance C*a* and centre C deſcribe a circle. Draw *a*O ⊥ to CB′ meeting the circle at O, and *a*I ⊥ to CA′ meeting the circle at I. Draw alſo OK ∥ to *a*I, ID ∥ to *a*O, DL ∥ to OK, KE ∥ to DI, EM ∥ to DL, FL ∥ to KE, and ſo on. Then becauſe of the parallels it is evident that

DI ... LF, GM, HN, &c. are ⊥ to CB′, and that

...M, FN, GQ, &c. are ⊥ to CA′;

the property of the circle,

am =

$am = mo$, $In = nD$, $Ko = oE$, $Lp = pF$, $Mq = qG$, $Nr = rH$, &c. and $ax = xI$, $Ow = wK$, $Dv = vL$, $Eu = uM$, $Ft = tN$, $Gs = sO$, &c.

Now it is well known that in a plain mirror the object appears to be as far behind as it is distant from the mirror, therefore O will be the place of the object a, as reflected from the mirror B′C, this image will be reflected again from O to K by the mirror A′C, then from K to E by the reflection of BC, and so on.

Again, I will be the place of the image of the object as reflected by A′C, this image will be reflected by B′C from I to D, by AC from D to L, and so on.

Therefore O, D, E, F, G, H, &c. will be the number of images of the object reflected by the mirror B′C, and I, K, L, M, N, O, &c, those reflected by the mirror A′C. Now when the object is placed on one of the mirrors, as A′C, the images at I, D, L, F, N, &c. will be wanting, and the others will appear as in Fig. 575, where it is evident that the arcs aO, aK, OE, KM, EG, NQ, are equal, each being equal to twice the given $\angle\, aCb = 30°$; hence $180 \div 30 = 6$, the number of images formed on the two mirrors, the last of which is imperfect, as it falls on the point of intersection of the mirrors. The distances Om, wK, oE, uM, and Gq, are easily determined, for they are respectively the sines of 15°, 30°, 45°, 60°, and 75°. Hence by *Page* 26, *of Emerson's Trigonometry*—$Om = 18\sqrt{(2 - \sqrt{3})}$, $wK = 18$, $oE = 18\sqrt{2}$, $uM = 18\sqrt{3}$, and $Gq = 18\sqrt{(2 + \sqrt{3})}$.

Messrs. Cunliffe, Gregory, *and* Lowry, *also favoured us with neat solutions to this question.*

VII. QUESTION 247, *answered by Mr.* William Francis, *Teacher of the Mathematics at Maidenhead.*

Plate 29, Fig. 576. Let ABCDE be the pentangular garden, and o the centre of its circumscribing circle. Let lines be drawn as in the figure, and draw the radius Ao which will be $\perp$ to EB at h. Now it is evident that the figure $cdeab$ will be a pentagon, and therefore the most regular inscribed curve will be a circle whose radius is oh. The $\angle\, hAE = 54°$ and $AE = 50$; hence Ah is found $= 29{\cdot}390$, and $hE = 40{\cdot}45085$: but 10 times hE is $=$ to the sum of all the walks $= 404{\cdot}5085$. Again, Ao is found $= 42{\cdot}5332$, and $Ao - Ah = oh = 13{\cdot}1432$ the radius of the circle, and its area $= 542{\cdot}697$ yards, which at 6d. per yard amounts to 13l. 11s. 4d.

The same, answered by Mr. Gregory, *the Proposer.*

It is plain that the angles BAC, and BCA are each $= 360 \div 10 = 36°$, and hence $ABC = 180 - 72 = 108°$. Therefore

as S. of 36° : S. of 108° :: AB (50) : 80·9016852 = AC = AD = BD = BE = EC; the sum of the lengths of these five walks is consequently = to 80·9016852 × 5 = 404·508426 yards. Now CdB = CAB + ABE = 36° × 2 = 72°, and CBd = 180° — (CdB + ACB) = 72°; therefore CdB is an isosceles triangle, and Cd = CB; in the same way it is shewn that Ab = AB or CB. Hence Ce = AC — BC = 30·9016852, and de = AC — 2Ce = 19·0983148. After a similar way we might find ae, ba, cb, or dc = 19·0983148. It is manifest that AdB = 108° — 36° × 2 = 108°; and of course the opposite angle bdc = 108°; thus it may be shewn that the angles dea, eab, abc, and bcd, are each = 108°—and, consequently the figure abcde having five equal sides, and five equal angles, is a regular pentagon: and the most regular inscribed curve will be a circle. Then, as S. of AoB : S. of ABo :: AB : Ao = 42·5882, also as, Rad. : AB :: S. of EBA : Ak = 29·390; hence Ao — Ak = 13·1432 the radius of the circle. The content of which is 542·697 square yards; and this at 6d. per yard amounts to 13l. 11s. 4d. the expence of digging the pond.

The same, answered by Tyro Philomatheticus.

Let ABCDE represent the garden, &c. Then it is evident that the pond must be a circle, the centre of which is the centre of the pentagon; it is likewise manifest that Dko is a right angle, and that Dk = kB, and from the nature of the figure oDk = 18° and kCD = 54°; therefore as, Rad. : DC (50) :: S. of kCD (54°) : Dk = 40·45085, and 40·45085 × 10 = 404·5085, the sum of the lengths of all the walks; and as, S. of Dok (72°) : Dk (40·45085) :: S. of oDk (18°) : ok = 13·1432 the radius of the pond; consequently its area is 542·697 square yards, which at 6d. per yard amounts to 13l. 11s. 4d. the expence of digging.

The same, answered by Mr. John Whitley.

Let ABCDEA represent the pentangular garden, AD, BE, AC, DB, and CE, the walks intersecting each other in the points *b*, *c*, *d*, *e*, *a*; then, by *Emerson's Geometry* IV. 48, DE*d*C, DEA*a*, EAB*a*, ABC*b*, and BCD*c*, are parallelograms, and therefore the angles C*d*E, CDE, D*e*A, DEA, &c. are equal, and consequently *abcdea* is a pentagon, and the curve which touches all its sides will be a circle, the centre of which will be, the centre of that circle which circumscribes the given pentagon. Let *o* be the centre and w *og* ⊥ to AD, and through *o* draw A*o*H meeting DC in H. en, by Cor. 3. of the above quoted proposition, we have as,

1:

$1 : (1 + \sqrt{5}) \div 2$ :: so is the side of the pentagon : to the diagonal, or the length of one of the walks; that is,
As $1 : (1 + \sqrt{5}) \div 2 :: DC (= 50) : DA = 25 (1 + \sqrt{5})$;
also, by Prop. 44, of the same Book,
As $5 - \sqrt{5} : 2 :: DC^2 : oA^2 = 2 \times 2500 \div (5 - \sqrt{5}) = 1250 + 250\sqrt{5}$;
and, by sim. Δs, DA : DH :: $oA : og$, that is,
$25(1 + \sqrt{5}) : 25 :: \sqrt{(1250 + 250\sqrt{5})} : \sqrt{(1250 + 250\sqrt{5})} \div (1 + \sqrt{5}) = og$ the radius of the circle touching the walks, the area of which is easily found $= 542 \cdot 7$ square yards nearly, which at 6d per yard comes to 13l. 11s. 4d. and the sum of the lengths of all the walks is equal to $(1 + \sqrt{5}) \times 25 \times 5 = 404 \cdot 5085$ yards.

W. W. R.

Ingenious solutions were also received from Messrs. Buffham, Lowry, May, *and* Thornoby.

VIII. QUESTION 248, *answered by Mr.* John Lowry.

The curve required is an exponential one, whose equation is $x = y^y$, where x represents the abscissa, and y the corresponding ordinate.

For if Y be put for the hyperbolic logarithm of y, then $x = y \times Y$ and $\dot{x} = \dot{y} \times Y + \dot{Y} \times y$; but by the nature of logarithms $\dot{Y} = \dot{y} \div y$; then by substitution $\dot{x} = \dot{y} \times Y + \dot{y}$, therefore $y\dot{x} \div \dot{y}$ (the general expression for the subtangent) $= y \times Y + y$: but $x = y \times Y$; therefore the subtangent $= x + y$, and the distance of the subtangent from the vertex is = to the ordinate y.

The same, answered by Mr. Marrat, *of Boston.*

If x denote the abscissa and y the ordinate of the curve required; then by the property of tangents, $\dot{y} : \dot{x} :: y :$ the subtangent, that is, by the nature of the question, $\dot{y} : \dot{x} :: y : y + x$, and by reduction $x\dot{y} - \dot{x}y + y\dot{y} = 0$. To find the fluent put $y = xz$, then $\dot{y} = \dot{z}x + z\dot{x}$, therefore by substitution we obtain $z^2x\dot{x} =$

$-x^2\dot{z} - x^2z\dot{z}$; therefore $\frac{\dot{x}}{x} = -\frac{\dot{z}}{z^2} - \frac{\dot{z}}{z}$, the fluent of which is,

hyp. log. $x = \frac{1}{z}$ — hyp. log. z: but $z = y \div x$; therefore

hyp. log. $x = \frac{x}{y}$ — hyp. log. $\frac{y}{x}$; hence $y \times$ hyp. log. $y = x$ and

consequently $y^y = x$, the equation of the curve required.

The same, answered by Tyro Philomatheticus.

By the question, $\frac{y\dot{x}}{\dot{y}} = x + y$, ($x$ being the abscissa, y the ordinate, and $\frac{y\dot{x}}{\dot{y}}$ the general expression for the subtangent); therefore to find the relation of x and y, suppose $zy = x$, then by substitution we get $y(z\dot{y} + y\dot{z}) = zy\dot{y} + y\dot{y}$, or $y\dot{z} = \dot{y}$, and $\dot{z} = \frac{\dot{y}}{y}$; hence $z =$ h. l. y, and the equation of the curve is $x = y \times$ h. l. y.

The same, answered by Mr. John Whitley.

Let $x =$ the abscissa, and $y =$ the ordinate of the required curve; then by the question $y = \frac{y\dot{x}}{\dot{y}} - x$; consequently $\frac{y\dot{x} - x\dot{y}}{y^2} = \frac{\dot{y}}{y}$; the fluent of which gives $\frac{x}{y} =$ hyp. log. y; whence $x = y \times$ hyp. log. y the equation of the curve required.

This question was likewise ingeniously answered by Messrs. Cunliffe, Gregory, *and* Thornoby.

IX. QUESTION 249, *answered by Mr.* Johnson, *of Birmingham.*

For the sum of the squares.

Construction. Pl. 30, Fig. 579. Join the three given point A, B, C; from C draw the diameter CSV, and let it be dividec in the given ratio at O. On OC as a diameter describe a semicircle, and from R, the middle of AB, apply RP to cut or touch the semi-circle, as at P, and such that $2AP^2 + 2RP^2 =$ the giver sum of th[...]es; then through P and C draw the chord CPD [...]

[...] it is done.

By Constr. CO : OV is = to the given ratio;

[...]I. Stone's Edition, CO : OV :: CP : PD;

[...]ided, at P, in the given ratio;

[...] = by constr. to the given sum of the squares

Fo[r]

For the difference of the squares.

Having as above joined the points A, B, C, drawn the diameter from C, and divided it in the given ratio at O, and on OC as a diameter described a semi-circle; then from R, the middle of AB, take RI such that AB $\times$ 2RI = the given difference of the squares, and at I erect the perpendicular IP to cut or touch the semi-circle at P, through which, and C, draw the chord CPD. Join AP, BP, and it is done.

The same, answered by Mr. John Lowry.

ANALYSIS. Pl. 30, Fig. 580. Suppose the chord CD to be really drawn as required. Join the points A and B, and draw the diameter CS, which divide at I in the given ratio. On CI as a diameter let a circle be described, and join IP, DS. Then, because of the right angles at P and D, the $\triangle$s ICP, SCD are similar; therefore the chord CD will always be divided in the given ratio by the circle CIP. Now when the *sum* of the squares of AP, BP is given, bisect AB at H, and join HP; then $AP^2 + BP^2 = 2AH^2 + 2PH^2$; but AH is given; therefore PH is given, and H is a given point, therefore the locus of the point P is a circle whose centre is H and radius HP, and consequently the intersection of this circle with the circle ICP will determine the point P.

Since AH is constant, the sum of the squares of AP, BP, will be the greatest or least possible when HP is the greatest or least, that is when it passes through the centre of the circle ICP.

Again when the *difference* of the squares AP, BP is given.

Draw PK $\perp$ to AB; then $AP^2 - BP^2 = AB \times 2KH$, therefore KH is given, and the locus of the point P is the straight line KP given by position; therefore its intersection with the circle ICP will determine the point P.

The difference of the squares will evidently be a maximum when KP touches the circle at P.

The same, answered by Mr. Swale.

Pl. 30, Fig. 581. Suppose the chord CD drawn as required, and divided in the given ratio at P. Join AB and bisect it at E, and join AP, BP, and EP. Then since $AP^2 + PB^2 = 2PE^2 + \frac{1}{2}AB^2$ is a given space, and AB a given line, it follows that PE is also given. Now join CE and draw DG parallel to PE meeting CE at G. By parallels PE : DG = CP : CD, a given ratio, or DG a given line: But, CG : CE = CD : CP, and therefore, because CE is given, the point G is a given one, that is, D is a given point.

Hence

Hence for the ſum of the ſquares.

Join the given points A, B, and biſect AB at E, and take the line EP ſuch, that $\frac{1}{2}AB^2 + 2PE^2 =$ the given ſpace; join CE and make EG to EC in the given ratio, and to the given circle, apply GD a 4th proportional to CE, CG, and EP; then join CD and it is done.

For the difference of the ſquares.

Demit upon AB the $\perp$ PF, join CF, and let DH, drawn parallel to PF, meet CF at H. Then ſince $AP^2 - PB^2 = AF^2 - FB^2 =$ a given ſpace, the point F will be given, ſince AB is a given line. Alſo, becauſe FH : FC = PD : PC, is a given ratio, the point H will be given. Whence in this caſe, join AB and divide it in F, ſo that $AF^2 - FB^2$ may be = to the given ſpace; join CF, and take FH : FC = the given ratio, and draw HI $\perp$ to AB, meeting the circle in D: then joining C, D, it is done.

Meſſrs. Cunliffe *and* Whitley *ſent neat ſolutions to this queſtion.*

X. QUESTION 250, *anſwered by Mr.* Cunliffe.

The area and the perimeter of the polygon being given the radius of the inſcribed circle becomes known; for the area is equal to a rectangle under half the perimeter and the radius of the inſcribed circle. Therefore let r denote the radius of the inſcribed circle, $2t$ one of the ſides of the polygon, n the number of its ſides, and p the given perimeter thereof. Let alſo, z denote the length of that arc of the inſcribed circle whoſe tangent is t, and put $c = 3 \cdot 14159$ &c.

Then it is obvious that $2nt = p$, and $2nz = (p \div t)\, z = 2rc$, the circumference of the inſcribed circle. Moreover it is well known that

$$z = t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^5} + \&c;\ \text{wherefore,}$$

$$2nz = (p \div t)\, z = (p \div t) \times (t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^5} + \&c.)$$

$$= p \times (1 - \frac{t^2}{3r^2} + \frac{t^4}{5r} - \frac{t^6}{7r^5} + \&c.)$$

$$= 2rc;\ \text{or}$$

$$1 - \frac{t^2}{3r^2} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6} + \&c. = 2rc \div p;\ \text{whence,}$$

$$t^2 - \frac{3t^4}{5r^2} + \frac{3t^6}{7r^4} - \frac{3t^8}{9r^6}\ \&c. = 3r^2 - \frac{6r^3c}{p} = q,\ \text{a given quantity,}$$

tity, becaufe both r and p are given; and by reverting the feries

$$t = q^{\frac{1}{2}} + \frac{3q^{\frac{3}{2}}}{10r^2} + \frac{141q^{\frac{5}{2}}}{1400r^4} + \frac{1411q^{\frac{7}{2}}}{42000r^6} + \&c.$$

$$= q^{\frac{1}{2}} \times (1 + \frac{3q}{10r^2} + \frac{141q^2}{1400r^4} + \frac{1411q^3}{42000r^6} + \&c.)$$

Now t being determined, n will be found from the equation $n = p \div 2t$.—In all cafes when p, the perimeter of the polygon does not much exceed $2rc$, the circumference of the infcribed circle, the preceding feries expreffing the value of t will converge faft, becaufe in thofe cafes the value of q will be but fmall. To prove this, put $2rc + v = p$, then

$$q = 3r^2 - \frac{6r^3c}{p} = 3r^2 - \frac{6r^3c}{2rc + v} = \frac{3r^2v}{2rc + v},$$ which, becaufe v is fmall, will evidently be fmall.

But the folution of any particular cafe of the queftion may likewife be eafily made out from what is done above by the method of trial and errour. For it has been found that $pz \div t = 2rc$, or $z \div t = 2rc \div p$, that is, the ratio of an arc of a circle to its tangent is given, from whence, by a few trials, the arc itfelf, as well as the tangent, may be found, and from thence n as before.

XI. QUESTION 251, *anfwered by Mr.* Lowry.

Plate 30, Fig. 582. Let CB be the given diftance of the illuminated point from the point directly under the candle, and BA the required height of the candle. Join CA, and from B upon CA drop the $\perp$ BD.

Now the effect of a particle of light to illuminate an object being directly as the fine of the angle of incidence, and reciprocally as the fquare of the diftance, or directly as BD, and inverfely as CA^2 we have $BD \div AC^2$ a maximum. But by fim. $\triangle$s $CA^2 : CB^2 :: CB^2 : CD^2$; therefore $BD \div AC^2 = (CD^2 \times BD) \div CB^4$; but CB^4 is a conftant quantity, therefore when $BD \div AC^2$ is a maximum, $CD^2 \times BD$ will be fo too; and by Simpfon on the max. et min. this will happen when $CD^2 = 2BD^2$, or when $CB^2 = 2BA^2$; therefore $BA : BC :: 1 : \sqrt{2}$, or as the fide of a fquare to its diagonal.

The fame, anfwered by Tyro Philomatheticus.

Let $a =$ the given diftance of the point to be illuminated from the point directly under the candle, and $x =$ the required altitude.

Then

Then it is well known that $\frac{1}{a^2+x^2}$ will be as the number of pari cles, and the force of each being as the fine of the angle of incidenc we have $\frac{x}{\sqrt{(a^2+x^2)}}$ as the force of one particle; therefore

$$\frac{x}{\sqrt{(a^2+x^2)}} \times \frac{1}{a^2+x^2} = \frac{x}{(a^2+x^2)^{\frac{3}{2}}},$$ is as the force or pow of the flame to illuminate the faid point, which by the queftioi muft be a *maximum*. Put into fluxions and reduced gives $x = a\sqrt{\frac{1}{2}}$.

XII. QUESTION 252, *anfwered by* Tyro Philomatheticus.

From a little confideration, it will manifeftly appear that th length of the curve will be found by the expreffion given for th involute of a circle; that is, putting $p = 3{\cdot}141592$ &c.

$$\left\{c^2 - \left(\frac{a}{2p} + \tfrac{1}{2}b\right)^2\right\} \div \left(\frac{a}{p} + b\right) = \text{the length of the curve}$$

XIII. QUESTION 253, *anfwered by Mr.* Lowry.

By the fcholium to Prop. XXXV, Emerfon's Mechanics " A body projected on an inclined plane will defcribe a parabola And if the velocity of projection upon the plane to be the velocit of a projectile in the air; as the relative gravity on the plane to th abfolute gravity; and both projected at the fame obliquity, the fam parabola will be defcribed in both cafes." Now the relative gravit on the plane is to the abfolute gravity, as the fine of the angl of elevation to radius, that is, as fine of 10° : radius :: 8 fee (the velocity of projection on the plane) to 46·0701 feet, th velocity neceffary to defcribe the fame parabola in free fpace; an $(46{\cdot}0701)^2 \div (4 \times 16\frac{1}{12}) = 32{\cdot}99151$ feet, the fpace whicl the ball muft defcend to acquire that velocity. Suppofe the plan of the table to be placed in a vertical direction, A the point fron whence the ball is projected, (Plate 30, Fig. 587) K the centre c the table, and F the place where the ball quits it. Draw KIS QABD, $\perp$ to the horizon, and take $AQ = QQ' = 32{\cdot}99151$ draw the indefinite right lines QSC, Q'C'C, and through A draw th tangent AC', to touch the circle at A and meet Q'C' at C', and fron C' to SK, apply C'u=C'A; then draw Au, and it is the directioi required, as is evident from Prop. IV. Page 37 and 38 of Burrow's Elegan

Elegant Little Essay on Projectiles; and by the rules there given the angle KAV is found $= 30° 41' 35''$. Again with the distances KS, AQ, and centres K, A, describe two arcs intersecting in P: then P will, by the same essay, be the focus of the parabola, and CQO its directrix.

Join AG, AK, KP, and KF, and draw OPN parallel, and AIL, KBM, and FEDN perpendicular to ADQ. Then by the property of the circle $GI = AG^2 \div 2GK = \cdot416'$ and $IR = 239\cdot583'$, $AI = 9\cdot992$, and by trigonometry $\angle AKI = 4° 46' 25''$. And in the Δ APK the three sides are given to find $\angle AKP = 3° 59'$; therefore the $\angle$ PKM is found $= 81° 14' 35''$, and KM and $PM = 78\cdot488$ and $509\cdot39$ respectively. Hence $KP - PM = PO = 6\cdot091 =$ half the latus rectum $= \frac{1}{2}p$.

Put $a = KM$ or HE, $b = KF$, and $x = FE$.

Then by the property of the parabola

$p : 2KM + FE :: FE : EK$, or

$p^2 : (2a + x)^2 :: x^2 : b^2 - x^2$; therefore

$p^2b^2 - p^2x^2 = 4a^2x^2 + 2ax^3 + x^4$, or

$x^4 + 2ax^3 + (4a^2 + p^2)x^2 = p^2b^2$, and from the resolution of this equation x is found $= 9\cdot027$, and $MN = 119\cdot66$. And by Dr. Hutton's Mensuration the arc HF is found $= 735\cdot083$, the arc $AH = 458\cdot767$, and their difference $AF = 276\cdot316$ the length of the track required.

XIV. QUESTION 254, *answered by* Collator, *the Proposer*.

Plate 30, Fig. 583. Let ABCD be the ellipse, F and f its foci, AC and BD its major and minor axes. Take any point E between A and B, join Ef, EF. Now, since heat is inversely as the square of the distance, the heat coming from F will be as $\frac{1}{(FE)^2}$, and the heat coming from f will be as $\frac{1}{(fE)^2}$; therefore the whole heat will be as their sum, or as $\frac{1}{(FE)^2} + \frac{1}{(fE)^2}$; but whilst FE is greater than fE the square of FE decreases faster than that of fE increases, because the sum of FE, and fE is always the same, and therefore FE decreases just as fast as fE increases; consequently $\frac{1}{(FE)^2}$ increases faster than $\frac{1}{(fE)^2}$ decreases. Hence,

whilst

whilst E is between A and B the sum $\frac{1}{(FE)^2} + \frac{1}{(fE)^2}$ increases as the point E approaches to B, and decreases as the same point recedes from B; consequently the heat will be a maximum at B, and a minimum at A.

The same, answered by Mr. Lowry.

Let AEBCD be the given ellipse, F and f the foci, and F any point in the curve. Join fE, FE. Then, since the effect of heat is reciprocally as the square of the distance, the heat at F will be as $\frac{1}{(FE)^2}$, and that at f as $\frac{1}{(fE)^2}$, and therefore the whole effect will be as $\frac{1}{(FE)^2} + \frac{1}{(fE)^2}$; and since CF + CG is given (= the transverse diameter AC) it is evident $\frac{1}{(FE)^2} + \frac{1}{(fE)^2}$ will be a maximum when FE = fE, and a minimum when fE exceeds FE by the greatest distance possible, or when fE coincides with AC; therefore the greatest heat will be at the extremity of the conjugate diameter, and the least heat at the extremity of the transverse diameter.

Solutions to this question were also received from Messrs. Francis *and* Tyro Philomatheticus.

XV. QUESTION 255, *answered by Mr.* Cunliffe.

Plate 30, Fig. 584. Let ACD be drawn through the centres C and O, of both circles; join DG and produce it till GM = GD. Draw ME to touch the circle whose centre is C in E; through C and E draw the chord CEP, upon which demit the ⊥ GV, and EP will be bisected in V.

For, draw MQ ∥ to EP, and meeting DP in Q, and produce VG to meet MQ in N. Then DP is at right angles to CP, therefore GN and DQ are parallel: but GM = DG by the construction, therefore MN = NQ, or their equals, EV = VP. *Q. E. D.*

If it had been required to draw the chord CP, such that demitting the ⊥ GV; EV and VP might have a given ratio, the construction would not have been materially different.

For

For in that case it had only been taking GM to DG in the given ratio, and the remaining part of the construction the same as before.

The same, answered by Mr. Johnson.

CONSTRUCTION. Plate 30, Fig. 585. On the line joining the points O, G, describe a semi-circle, and apply therein OK equal to half the radius of the circle whose centre is C; draw CP $\parallel$ to OK and it is done.

DEMONSTRATION. Draw OI $\perp$ to CP, and produce GK to meet CP at L; then we have to prove that $EL = LP$. Now by the property of the circle $CI = IP$, or $CE + EI = IL + LP$, and by construction $2OK$ or $2IL = CE$; therefore $2IL + EI = IL + LP$, and taking IL from each, there remains $EL = LP$. *Q. E. D.*

If $2OG$ be less than CE the problem is impossible.

The same, answered by Mr. Swale, *the Proposer.*

CONSTRUCTION. Plate 30, Fig. 586. Join CG, OG, and in OG take the point L such that $GL^2 - LO^2 =$ the difference between the square of CG and the sum of the squares of the radii CO, CA; draw LE $\perp$ to GO meeting the former circle in E; and through C and E draw CP and it is done.

DEMONSTRATION. By the construction,

$GL^2 - LO^2 = CG^2 - CO^2 - CA^2 = GE^2 - EO^2$, that is,

$GE^2 + EC^2 - EO^2 = CG^2 - CO^2 = CG^2 - OI^2$.

But, because GF is perpendicular to CP, we have,

$CG^2 = GE^2 + EC^2 + CE \times 2EF$; or

$GE^2 + EC^2 - EO^2 = GE^2 + EC^2 + CE \times 2EF - OI^2$:

Then, $OI^2 - EO^2 = CE \times 2EF$.

Now, since $HO = OI$, it follows that,

$OI^2 - OE^2 = EI \times EH$; Hence

$CE \times 2EF = EI \times EH = CE \times EP$, by the circle;

Then $2EF = EP$, or $EF = FP$;

that is EP is bisected at F by the $\perp$ GF. *Q. E. D.*

Ingenious solutions were also given by Messrs. Lowry *and* Whitley.

XVI. QUESTION 256, *answered by Mr.* John Whitley.

Plate 30, Fig. 588. Let lines be drawn as directed by the question, and join OB, OC, GC. By known properties

AC²

$$
\begin{aligned}
AC^2 + CB^2 &= 2GB^2 + 2GC^2 \\
&= 2GB^2 + 2GO^2 + 2CO^2 + 4GO \cdot OH, \\
&\quad (\text{because } GC^2 = GO^2 + OC^2 + 2GO \cdot OH) \\
&= 2GB^2 + 2BO^2 + 2CO^2 + 4RO^2 \\
&= 2BO^2 + 2CO^2 + 4RO^2 \\
&= 4FO^2 + 4RO^2 \\
&= 4FR^2.
\end{aligned}
$$

Q. E. D.

The same, answered by Tyro Philomatheticus.

Draw FD $\perp$ to CA and join AE. Then CD = half the sum of the sides; and by Simpson's Trigonometry
$EG^2 : CD^2 :: AE^2 (= EG \cdot EF) : CF^2 (= FH \cdot EF) : EG : FH$, $\therefore$
$FH \cdot EG^2 = CD^2 \cdot EG$, or $FH \cdot EG = CD^2$: but
$CD^2 = \frac{1}{4}(AC^2 + 2AC \cdot BC + BC^2)$. Hence

$$
\begin{aligned}
FH \cdot EG &= \tfrac{1}{4}(AC^2 + 2AC \cdot BC + BC^2), \text{ that is,} \\
FH \cdot EG - \tfrac{1}{2}AC \cdot BC &= \tfrac{1}{4}(AC^2 + BC^2), \text{ or} \\
FH \cdot EG - FO \cdot GH &= \tfrac{1}{4}(AC^2 + BC^2), \text{ or} \\
(FG + GH)(FO + GO) - FO \cdot GH &= \tfrac{1}{4}(AC^2 + BC^2), \text{ or} \\
(FG + GH) \cdot GO + FG \cdot FO &= \tfrac{1}{4}(AC^2 + BC^2), \text{ or} \\
(FG + GO) \cdot FO + GO \cdot OH &= \tfrac{1}{4}(AC^2 + BC^2), \text{ or} \\
FO^2 + OR^2 &= \tfrac{1}{4}(AC^2 + BC^2), \text{ or} \\
FR^2 &= \tfrac{1}{4}(AC^2 + BC^2).
\end{aligned}
$$

Therefore $4FR^2 = AC^2 + BC^2$. *Q. E. D.*

The same, answered by Mr. Johnson, *Birmingham.*

By Euc. I, 47, $FO^2 + OR^2 = RF^2$, or $4FO^2 + 4RO^2 = 4RF^2$.
But $4RO^2 = 4OG \cdot HO$, and $4FO^2 = 4AO^2 = 4AG^2 + 4GO^2$.
Therefore $4AG^2 + 4GO^2 + 4OG \cdot HO = 4FR^2$.
But, Euc. III. 33, $4GO^2 + 4HO \cdot GO = 4GO \cdot GH$;
Hence $4AG^2 + 4GO \cdot GH = 4FR^2$.
But, Simp. Trig. $2AG^2 + 4GO \cdot GH = 2CG^2$.
Therefore $2AG^2 + 2CG^2 = 4FR^2$.
But, Simpson's Geometry $2AG^2 + 2CG^2 = AC^2 + CB^2$.
Therefore $4FR^2 = AC^2 + CB^2$.

Q. E. D.

Neat demonstrations were also received from Messrs. Lowry *and* Swale.

XVII. QUESTION 257, *answered by Mr.* Lowry.

Plate 30, Fig. 589. Let ABCH be the vessel and ID the rod meeting

meeting the surface of the water at S. Draw DEF ⊥ to the horizon, join ES, and draw SG ⊥ to ES meeting EF produced at G. Then, by the principles of optics, the rod ISD will appear in the position ISE, and ES^2 to DS^2 will be in the constant ratio of 16 to 9. Now it is evident that the apparent bending of the rod will be a maximum, when the angle ESD is so. Conceive a segment of a circle to be described through the points E, D, B; then it is well known that the angle ESD will be a maximum when AB touches the circle at S; and then

$FS^2 = FD \times FE = GF \times FE$, therefore

$FG = FD$ and $SG = SD$, and therefore

$SE^2 : SD^2 :: FE : GF :: 16 : 9$, or

$FE^2 : GF \times FE :: 16 : 9$, that is,

$FE : FS :: 4 : 3$.

Hence the angle DSE $= 36^\circ\ 52'\ 12''$, the angle required.

XVIII. QUESTION 258, *answered by* Northumbriensis.

In the consideration of this question we may divide it into two cases.

1. When the latitude is greater than the sun's declination; and
2. When it is less than the declination.

In the first case it has been shewn (See Mr. *Gregory's Astronomy* art. 676) that the variation of the sun's altitude is greatest when on the *prime vertical*: and the time when this is the case may be found by the well known proportion, as rad. : tang. declination :: cotang. latitude : cosine hour angle from noon. But, in the second case, the variation is the greatest, when the sun's azimuth is a *maximum*, (art. 181) and the time of this may be found by saying, as rad. : cotang. declination :: tang. latitude : cosine hour angle from noon.

In the consideration of both these cases no regard is paid to the effects of refraction, or to the change in the sun's declination.

XIX. QUESTION 259, *answered by Mr.* Cunliffe.

Let the borings at A, B, and C, (Fig. 590, Pl. 30.) be represented by the lines AA′, BB′, and CC′ at right angles to the horizontal plane ABC, and imagine another plane to pass through the points A′, B′, C′, and to be continued till it intersects the horizontal plane in the line DE; then, this latter plane will manifestly represent the plane of the stratum of coal; the direction and dip of which may be found as follows.

Draw

Draw AF parallel to DB, and CG parallel to EB, each terminating in BB′; alſo upon ED let fall the perpendicular BH.

By the queſtion, AB = A′F = a; BC = C′G = c; AA′ = BF = p; CC′ = BG = r; whence B′F = $q - p$ and B′G = $q - r$; and becauſe of the parallels A′F, DB.

B′F : A′F :: B′B : D′B = (A′F · B′B) ÷ B′F = $aq \div (q - p)$.

Alſo becauſe of the parallels C′G, EB,

B′G : C′G :: BB : EB = (C′G · B′B) ÷ B′G = $cq \div (q - r)$.

Now in the △ ABC all the ſides are given; whence the ∠ ABC may be found by trigonometry. Moreover in the △ DBE, the ſides DB and EB, together with their included ∠ DBE, are given; to find the ∠ EDB or its ſupplement, the ∠ HDB; and from thence the complement of this, or the ∠ DBH becomes known.

Suppoſe BM to be a meridian paſſing through B: the ∠ ABM is given by the queſtion; therefore, as the ∠ DBH is given, the ∠ HBM becomes known, or the angle becomes known which the direction of the dip of the ſtratum makes with the meridian. Now HB may be eaſily determined from what has already been ſhewn: and at the length HB the ſtratum manifeſtly dips the depth BB′.

XX. QUESTION 260, *anſwered by* Collator, *the Propoſer.*

Let A be the weight of the ſyſtem, a the radius of the cylinder, x the diſtance of the centre of gyration from the centre of gravity, $m = 16\frac{1}{12}$, V the velocity generated in the time T, and S the ſpace paſſed over by the weight P.

Now, by the property of the centre of gyration, if all the matter in the ſyſtem were collected, and uniformly diſpoſed over the circumference of a circle whoſe radius is x, and whoſe plane is perpendicular to the axis of motion, and paſſing through the centre of gravity of the ſyſtem; the ſame force would generate the ſame angular velocity in the ſame time: alſo if inſtead of the matter being diſpoſed at the diſtance of the centre of gyration, an equivalent maſs be ſubſtituted at the ſurface of the cylinder, the ſame angular velocity will ſtill be preſerved; which maſs will be expreſſed by $\frac{x^2A}{a^2}$.

By Atwood on *Rectilinear Motion and Rotation*, p. 36, $\frac{Q}{q} = \frac{M}{m} \times \frac{v}{V}$; and, in this caſe, M = m = P; therefore $\frac{Q}{q} = \frac{v}{V}$, and

and $Q = N = \frac{Ax^2}{a^2}$; and the whole quantity of matter, moreover, $= \frac{Ax^2}{a^2} + P = \frac{Ax^2 + Pa^2}{a^2} = Q$; but $\frac{Q}{q} = \frac{v}{V} = \frac{2m}{V}$, and $q = P$, therefore $\frac{Q}{q} = \frac{Q}{P} = \frac{Ax^2 + Pa^2}{Pa^2} = \frac{2m}{V}$.

Hence $V = \frac{2m\,Pa^2}{Ax^2 + Pa^2} = 2\sqrt{(mS)}$. Therefore

$2\sqrt{(mS)} \times Ax^2 + 2\sqrt{(mS)} \times Pa^2 = 2m\,Pa^2$, or
$2\sqrt{(mS)} \times Ax^2 = 2mPa^2 - 2\sqrt{(mS)} \times Pa^2$, and

$$x^2 = \frac{2mPa^2 - 2\sqrt{(mS)} \times Pa^2}{2\sqrt{(mS)} \times A} = \frac{2Pa^2 \times \left\{m - \sqrt{(mS)}\right\}}{2\sqrt{(mS)} \times A}.$$

Confequently $x = \pm \sqrt{\frac{2Pa^2 \times \left\{m - \sqrt{(mS)}\right\}}{2\sqrt{(mS)} \times A}}$, the diftance of gyration from the centre of gravity; the latter of which being determinable by well known methods, it is unneceffary to purfue the inveftigation further.

XXI. QUESTION 261, *anfwered by Mr.* Cunliffe, *the Propofer.*

Fig. 591, Pl. 30. Let DBA reprefent the catenarian curve AD its axis, and DB its greateft ordinate, about which it revolves in the generation of the folid.

Draw any ordinate QP, and alfo the line PR parallel to AD meeting DB in R. Put $l =$ the length of the curve APB, AD $= b$, AQ $= x$, QP $= y$, $a =$ the tenfion of the curve at A, and $p = 3{\cdot}1415926$, Then by the property of the curve

$\dot{y} = \frac{a\dot{x}}{\sqrt{(2ax + x^2)}}$;—and by the writers on fluxions

$$(RP)^2 \times p \times \text{flux.}DR = \frac{p \times (b - x)^2 \times a\dot{x}}{\sqrt{(2ax + x^2)}}$$

$$= ap \times \left(\frac{b^2\dot{x}}{\sqrt{(2ax+x^2)}} - \frac{2bx\dot{x}}{\sqrt{(2ax+x^2)}} + \frac{x^2\dot{x}}{\sqrt{(2ax+x^2)}}\right)$$

will exprefs the fluxion of the folid generated by the revolution of the fpace DAPR about the ordinate DB: the fluent of which is

$$ap \times \left(\frac{x}{2}\sqrt{(2ax + x^2)} - (\frac{3a}{2} + 2b)\sqrt{(2ax + x^2)}\right)$$

$$+ ap \times (b^2 + \frac{3a^2}{2} + 2ab) \times \text{h. l.}\left(\frac{a + x + \sqrt{(2ax + x^2)}}{a}\right);$$

and this expreſſion is correct, for the whole will vaniſh or become $= 0$, when $x = 0$, as it ought.

But when $x = b$, the expreſſion becomes

$$ap \times (b^2 + \frac{3a^2}{2} + 2ab) \times \text{hyp. log.}\left(\frac{a + b + \sqrt{(2ab + b^2)}}{a}\right)$$

$-\, ap \times (\frac{3a + 3b}{2})\sqrt{(2ab + b^2)}$ for the whole content of the the ſolid generated by the revolution of the ſpace DPBA about the ordinate DB.

Again, by the property of the curve $a = (l^2 - b^2) \div 2b$; let this be written inſtead of a in the preceding expreſſion for the ſolidity and it becomes

$$p \times (\frac{l^2 - b^2}{2b}) \times (\frac{3(l^2 - b^2)^2}{8b^2} + l^2) \times \text{h. l.}(\frac{l+b}{l-b}) - \frac{3lp}{8} \times (\frac{l^4 - b^4}{b^2})$$

which is to be a *maximum* by the queſtion, or

$$\left\{\tfrac{3}{2}(\frac{l^2 - b^2}{2b})^3 + l^2\,(\frac{l^2 - b^2}{2b})\right\} \times \text{h. l.}(\frac{l+b}{l-b}) - \frac{3l}{2}\,(\frac{l^4 - b^4}{4b^2})$$

muſt be a maximum: therefore conſidering b variable and putting the fluxion of the expreſſion $= 0$, there will be had

$$\left.\begin{aligned}&-\dot{b}\,(\frac{l^2 + b^2}{2b^2}) \times \left\{\tfrac{9}{2} \times (\frac{l^2 - b^2}{2b})^2 + l^2\right\} \times \text{h. l.}(\frac{l+b}{l-b})\\ &+ l\dot{b} \times (\frac{3(l^2 - b^2)^2}{8b^3} + \frac{l^2}{b}) + 3l\dot{b} \times (\frac{l^4 + b^4}{4b^3})\end{aligned}\right\} = 0;$$

and dividing by $\frac{\dot{b}}{b^2}$

$$\left.\begin{aligned}&-(\frac{l^2 + b^2}{2}) \times \left\{\tfrac{9}{2} \times \left(\frac{l^2 - b^2}{2b}\right)^2 + l^2\right\} \times \text{h.l.}(\frac{l+b}{l-b})\\ &+ l \times \left\{\frac{3(l^2 - b^2)^2}{8b} + l^2 b\right\} + 3l\,(\frac{l^4 + b^4}{4b})\end{aligned}\right\} = 0;\ \text{or}$$

$$\left.\begin{aligned}&-(\frac{l^2 + b^2}{2}) \times (\frac{9l^4 - 10l^2b^2 + 9b^4}{8b^2} \times \text{h. l.}(\frac{l+b}{l-b})\\ &+ lb \times (\frac{9l^4 + 2l^2b^2 + 9b^4}{8b^2})\end{aligned}\right\} = 0.$$

From

From whence h. l. $\left(\frac{l+b}{l-b}\right) = \frac{2lb \times (9l^4 + 2l^2b^2 + 9b^4)}{(l^2+b^2) \times (9l^4 - 10l^2b^2 + 9b^4)}$, which expression is well adapted for finding the value of b by the method of trial and error; and by a few approximations of this sort we get $b = l \times \cdot 7864616$, which is correct to the last decimal place.

Again, from the property of the curve,

$$DB = \left(\frac{l^2-b^2}{2b}\right) \times \text{h. l.} \left(\frac{l+b}{l-b}\right) = l \times \cdot 515172333, \text{ or}$$

$$CB = l \times 1 \cdot 030344666.$$

W. W. R.

☞ All the properties of the catenary mentioned in the solution to this and question 265 may be seen, together with their investigations, in Dr. Hutton's Mathematical Dictionary.

XXIII. QUESTION 263, *answered by* Collator, *the Proposer.*

Plate 31, Fig, 603. Let ADB be a great circle of the earth or the equator; C its centre; AF the spiral. Put $r =$ AC $=$ rad.; $m = 16\frac{1}{12}$ feet; and the force of gravity at the surface of the earth $= 1$.

The velocity acquired by a body in falling through $\frac{1}{2}r$, by the force of gravity, will be $=$ the velocity with which a body may describe a circle at the distance AC; therefore

$$\sqrt{m} : \sqrt{\frac{r}{2}} :: 2m : \frac{2m}{\sqrt{m}} \times \sqrt{\frac{r}{2}} = \sqrt{(2mr)} = \text{the velocity in}$$

the circle ADB per second.

Let n be the number of feet in the circumference of the earth, then $\sqrt{(2mr)} : n :: 1 : n \div \sqrt{(2mr)} =$ period. time in the circle ADB, which put $= p$. Let P be the period. time of the earth's revolution, and since C, the centrifugal force, is as $\frac{r}{p^2}$, and r is here given, C is as $\frac{1}{p^2}$, $P^2 : p^2 :: 1 : p^2 \div P^2 = a$, the centrifugal force at A. Again (if $x =$ dist. CF) since C is as $\frac{x}{p^2}$, and p is given, C is as x, and therefore $r : x :: 1 : \frac{x}{r} =$ the centripeta

force

force at F; also $r : x :: a : \frac{ax}{r}$ = the centrifugal force at F, whence the effective force tending to the centre C $= \frac{x}{r} - \frac{ax}{r} = \frac{x}{r} \times (1-a) = x \times \frac{1-a}{r} = F$. Now, from the general equation, $F\dot{x} = v\dot{v}$, we may derive a value for v in terms of x, for by substitution, $x\dot{x} \times \frac{1-a}{r} = v\dot{v}$, and $x^2 \times \frac{1-a}{r} = v^2$; and therefore $v = x\sqrt{\frac{1-a}{r}}$: whence v is as x, and therefore the spiral is the logarithmic one.

XXIV. QUESTION 264, *answered by Mr. Lowry.*

Pl. 30, Fig. 602. Let A, B be the given points, EF the given chord, and O the centre of the given circle. About the centre O describe a circle to touch EF at G, then OG being drawn will be $\perp$ to EF, and of a given length; suppose the $\perp$s AC, BD to be drawn as required. Then,

First. *When the ratio of BD to AC is given.*

Let EF and BA be produced till they meet at H.

Because AC : BD :: AH : AB, it follows, by division, that the ratio of AH : AB is given, and AB is given in magnitude, therefore AH is given, and H is a given point. Consequently if HEF be drawn to touch the circle at H, the thing is done.

Secondly. *When the sum or difference of AC and BD are given.*

These cases are constructed generally at Article 28, Vol. I. of the Repository, and a reference to that article is all that is necessary in this place.

Thirdly. *When the rectangle BD · AC is given.*

Through the centre O draw IOS parallel to EF, meeting AC in I, and BD in S; join AO and draw OK $\perp$ to AB, KT $\perp$ to OI, KL $\perp$ to AO, and AV $\perp$ to KT. Then

$BD = BS + OG$ and $AC = AI + OG$, therefore

$BD \cdot AC = BS \cdot AI + OG \cdot (BS + AI) + OG^2$, and therefore

$BS \cdot AI + OG \cdot (BS + AI) +$ is equal to a given space,

equal, suppose, to $AO \cdot AZ$. Now

TK is evidently equal to half the sum of BS, AI, and

KV

KV equal to half their difference; therefore
$BS = TK + KV$, and $AI = TK - KV$; wherefore
$BS \cdot AI = TK^2 - KV^2$ and $TK^2 - KV^2 + 2OG \cdot TK = AO \cdot AZ$, or
$LO \cdot TK^2 - LO \cdot KV^2 + 2OG \cdot LO \cdot TK = AO \cdot LO \cdot AZ$:

But the triangles AKV, KOT are similar, therefore
$AL : LO :: AK^2 : KT^2 :: KV^2 : TO^2$, or
$$LO \cdot KV^2 = AL \cdot TO^2 = AL \cdot (OK^2 - KT^2)$$
$$= AL \cdot LO \cdot OL - AL \cdot KT^2, \text{ wherefore}$$
$$\left.\begin{array}{l} LO \cdot TK^2 + AL \cdot TK^2 \\ \quad + 2OG \cdot LO \cdot TK \end{array}\right\} = AO \cdot LO \cdot AZ + AL \cdot AO \cdot OL, \text{ or}$$
$AO \cdot TK^2 + 2OG \cdot LO \cdot TK = AO \cdot LO \cdot LZ$,
Make $AO \cdot OQ = 2OG \cdot LO$, then
$AO^2 + TK^2 + AO \cdot OQ \cdot TK = AO \cdot LO \cdot LZ$, or
$TK^2 + OQ \cdot TK = LO \cdot LZ$, therefore TK becomes known by the 29th. VI. of the Elements, (See Playfair's Edition,) and the method of construction is obvious.

Fourthly. *When the sum of the squares of BD, AC is given.*
$BD^2 + AC^2 = BS^2 + AI^2 + 4OG \cdot KT + 2OG^2$,
therefore $BS^2 + AI^2 + 4OG \cdot KT$ is given:
but $BS = TK + VK$, and $AI = TK - KV$,
therefore $BS^2 + AI^2 = 2TK^2 + 2KV^2$, and consequently
$TK^2 + KV^2 + 2OG \cdot KT$ is given, $=$ (suppose,) to $AO \cdot AM$,
that is, $LO \cdot TK^2 + LO \cdot KV^2 + 2OG \cdot LO \cdot KT = AO \cdot AM \cdot LO$.
But by similar triangles,
$AL : LO :: AK^2 : KO^2 :: KV^2 : TO^2 = OK^2 - TK^2$;
therefore $LO \cdot VK^2 = AL \cdot (OK^2 - TK^2)$, wherefore
$$\left.\begin{array}{l} LO \cdot TK^2 + AL \cdot (OK^2 - TK^2) \\ \qquad + 2OG \cdot LO \cdot KT \end{array}\right\} = AO \cdot AM \cdot LO, \text{ or}$$
$$(LO - AL) \cdot TK^2 + 2OG \cdot LO \cdot KT = AO \cdot AM \cdot LO - AL \cdot KO^2$$
$$= AO \cdot LO \cdot LM.$$
Make $2OG \cdot LO = (LO - AL) \cdot OX$,
and $LO \cdot LM = (LO - AL) \cdot OY$.
Hence $TK^2 + OX \cdot TK = AO \cdot OY$.

Consequently TK may be determined as before.

Finally. *When the difference of the squares of BD, AC is given.*
$$BD^2 - AC^2 = BS^2 - AI^2 + 2OG \cdot (BS - AI)$$
$$= 4TK \cdot VK + 4OG \cdot VK;$$
But the ratio of OT to VK is given;
therefore $(TK + OG) \cdot OT$ is given;
Draw RW $\perp$ to KO produced at W; then
$OK : OT :: KR \cdot (TK + OG) : RW$, or
$OK \cdot RW = (TK + OG) \cdot OT$; therefore RW is a given line, and the *locus* of the point R is a straight line RU drawn parallel to KO. Also the locus of the point T is a semi-circle described on OK; it is therefore required to draw KR so that the inter-

cepted part TR may be of a given length, which is a solid problem, and may be effected by the Conic Sections, or (much easier) by the conchoidial ruler.

XXV. QUESTION 265, *answered by Mr.* Cunliffe, *the Proposer.*

Fig. 592, Pl. 30. Let APB represent a position of the chain, and imagine SV to be drawn $\perp$ to the middle of AB, meeting the curve of the chain in V: also let PQ and PR be drawn $\perp$ to SV and AB respectively.

Put $2l =$ APB the given length of the chain, or $l =$ APV, $m =$ the length of the part PV, $SV = b$, $AR = x$, $RP = SQ = y$; also let a denote the tension of the chain at V when in the position AVB.

Then $VQ = b - y$, and by the property of the curve,

$$a = \frac{m^2 - (b-y)^2}{2(b-y)} = \frac{l^2 - b^2}{2b}, \text{ whence } bm^2 + b^2y - by^2 = bl^2 - l^2y,$$

which being properly ordered gives

$$b = \frac{l^2 + y^2 - m^2}{2y} - \frac{\sqrt{\left\{(l^2 - y^2) - 2m^2(l^2 + y^2) + m^4\right\}}}{2y}.$$

Again, by the property of the curve,

$$AS = a \times \text{h.l.}\left(\frac{AV + SV}{AV - SV}\right) = a \times \text{h.l.}\left(\frac{l+b}{l-b}\right), \text{ and}$$

$$PQ = RS = a \times \text{h.l.}\left(\frac{PV + QV}{PV - QV}\right) = a \times \text{h.l.}\left(\frac{m+b-y}{m+y-b}\right); \text{ whence}$$

$$AS - RS = AR = x = a \times \left\{\text{h.l.}\left(\frac{l+b}{l-b}\right) - \text{h.l.}\left(\frac{m+b-y}{m+y-b}\right)\right\}$$

$$= a \times \text{h.l.}\frac{(l+b)\times(m+y-b)}{(l-b)\times(m+b-y)}$$

$$= \frac{l^2 - b^2}{2b} \times \text{h.l.}\frac{(l+b)\times(m+y-b)}{(l-b)\times(m+b-y)},$$

by substituting for a its equal $\frac{l^2 - l^2}{2b}$.

And exterminating b from this equation by means of its value found above, there will be had an equation expressing the nature of the

the curve, which is the locus of the point P, in terms of its own abſciſſa and ordinate.

The area of the curve may be generally expreſſed by ſeries.

XXVI. QUESTION 266, *anſwered by Mr.* Lowry.

Plate 30, Fig. 593, 594. ANALYSIS. Suppoſe it done, and let A and O be the centres of the given circles, B the given point in the line joining their centres, D the point of contact, and C the centre of the required circle. Draw CA and CB; then CA will paſs through D, the point of contact, and CD will be = CB. Therefore AC + CB = AD in fig. 593, and AC — CB = AD in fig. 594, = to a given line.

Draw the radius CO, then (Stewart's General Theorems, Prop. II.) AC^2 together with the ſpace to which BC^2 has the ſame ratio that BO has to AO, is equal to the rectangle BAO together with the ſpace to which CO^2 has the ſame ratio that BO has to AB, that is equal to a given ſpace, equal to P^2.

Therefore the problem is reduced to the following, viz. Given the ſum or difference of two lines to determine them, when the ſquare of one, together with the ſpace to which the ſquare of the other has a given ratio, is equal to a given ſpace; which may be effected as follows:

Take ad = the given ſum or difference (ſee fig. 595 and 596), and on ad (produced when the difference is given) take any point b and erect the perpendicular bg ſo that $db^2 : bg^2$ in the given ratio of BO : AO; draw dg and from a apply am = to P; draw alſo $mc \perp$ to ad; ſo ſhall ac, cd be the two lines required; that is, if AC be made = ac, the point C will be determined.

For cm^2 is evidently the ſpace to which cd^2 or CB^2 has the ſame ratio that BO has to AO, and $ac^2 + cm^2 = AC^2 + cm^2 = P^2$ = the given ſpace.

XXVII. QUESTION 267, *anſwered by Mr.* Lowry, *the Propoſer.*

Plate 30, Fig 597. CONSTRUCTION. Biſect the given baſe AB at D, and draw DO ⊥ to AB, and ſuch that 4DO may be to DB in the given ratio; on DO deſcribe a ſemicircle, and therein apply DC = DB; join AC, BC, and ACB is the △ required.

About the centre, with the diſtance OCO, deſcribe a circle cutting DO produced in E and F. Then we ſhall prove,

1ſt. That the circle ECF is the locus of the vertices of all the

△s

$\triangle$s constituted on the base AB when the sum of the squares of the sides has to the area of the $\triangle$ the ratio of 4DO to DB.

2ndly. That C is the point in the circumference ECF where the ratio of AC to BC is the greatest possible.

Let C′ be any point in the circumference; join AC′, BC′, OC′, and DC′, and draw CH parallel, and C′G perpendicular to AB. Then

$OD^2 - DB^2 = OC^2$ or $C'O^2 = C'H^2 + HO^2$, or

$OD^2 - OH^2 (= 2OH \cdot HD + HD^2) = C'H^2 + DB^2$,

and adding HD^2 to each,

$2OD \cdot HD = C'H^2 + DB^2 + HD^2 = DB^2 + C'D^2$; therefore

$4OD \cdot HD = 2DB^2 + 2C'D^2 = C'A^2 + C'B^2$. Consequently

$C'A^2 + C'B^2 (4OD \cdot HD)$: area of the $\triangle AC'B (BD \cdot DH) :: 4OD : DB$.

And if the points C′ and C coincide

$AC^2 + BC^2$: area $\triangle$ ACB :: 4OD : DB, that is, in the given ratio.

Again, produce OC to meet AB produced in S, then since AD = DB = DC, and the $\angle$DCS a right one, CS is a tangent to the circle which passes through the points A, C, B. With the centre S, and distance SC describe a circle, cutting BC′, in c, and join Ac.

Then $SC^2 = BS \cdot SA$, and by a well known proposition

AC : CB :: Ac : Bc; but

the ratio of Ac : Bc is evidently greater than that of AC : BC; therefore the ratio of AC : CB is greater than that of AC′ : BC′.

The triangle ACB is right angled at C.

XXVIII. QUESTION 268, *answered by Mr.* Lowry.

Plate 30, Fig. 598. CONSTRUCTION. On the given line AD describe a segment of a circle to contain an angle equal to half the given vertical angle. At A draw a tangent AI of such a length that its square may be equal to the given difference. Through the point I describe a circle concentric with the other, and from B, the extremity of the diameter drawn perpendicular to AD, draw BC to make the $\angle$ BCA = the given one; shall ACB be the $\triangle$ required.

Take CF = BC, and join BF, DE, and from B draw the tangent BH. Then since CF = BC, AF is equal to the sum of the sides of the $\triangle$ ACB, and the $\angle$ CFB = CBF = half the vertical $\angle$ ACB, =, by construction, to the $\angle$ AED, and the $\angle$ EAD is common to the $\triangle$s ABF, ADE; therefore they are similar, and

$$AD \times AF = AE \times AB: \text{ but,}$$

$$AI^2 = BH^2 = BE \times AB; \text{ therefore}$$

$$AD \times AF + AI^2 = AE \times AB + BE \times AB = AB^2.$$

Therefore

Therefore the ſquare of the baſe exceeds the rectangle under the ſum of the ſides and and a given line, by the given difference.

The perpendicular BK is evidently the greateſt that can be inſcribed in the circle when it paſſes through the centre, and therefore in this caſe muſt be the greateſt poſſible.

XXIX. QUESTION 269, *anſwered by Mr.* Lowry, *the Propoſer.*

This problem admits of three caſes according to the ſituation of the given points with reſpect to circle; viz.

1ſt. When the given points are without the circle.

2ndly. When two of them are within, and the remaining one without the circle; and

3rdly. When one of the points is within and the other two without the circle.

The conſtruction for the firſt caſe is as follows. Fig. 599, Pl. 30.

Let A, B, C, be the three given points, KHLM the circle in which the trapezium is to be inſcribed, and IS the other given circle. Draw the tangents AD, BE, and in AB aſſume the point F (on the ſame ſide of A with the point C) ſo that the rectangle CAF $= AD^2$, and in AB find the point G (on the ſame ſide of B with the point F) ſo that the rectangle GBF $= BE^2$; draw GM to touch the given circle IS at I, alſo draw CM meeting the circle at K, AK meeting the circle at H, BH meeting the circle at L, join LM, and HKLM is the trapezium required.

Again, for the 2nd and 3rd Caſes. Fig. 600, 601, Pl. 30.

In AB aſſume the point F ſo that the rectangle CAF $= AD^2$, and the point G ſo that the rectangle GBF $=$ PBQ; draw GM to touch the given circle IS at I, alſo draw CM meeting the circle at K, AK meeting the circle at H, BH meeting the circle at L, join LM, and HKLM is the trapezium required.

For, by Prop. XLV. and XLVI. Lib. I, of Dr. *Stewart's Propoſitiones Geometricæ*, the points L, M, G are in a ſtraight line, therefore the ſides HK. KL, and KM paſs through the given points A, B, C, reſpectively, and the ſide MI. touches the given circle.

Q. E. D.

XXX.

XXX. Or, PRIZE QUESTION 270, *anſwered by the Propoſer*, Philalethes Cantabrigienſis.

In printing this queſtion, the ſine + was put inſtead of the ſine — before the term $2\sqrt[3]{x}$, (as was noticed on the cover of No. XII.) which will occaſion a great difference in the value of y, and incumber the numerical calculation with decimals, but the method of ſolution will be nearly the ſame. I ſhall therefore give a ſolution to the queſtion as it was written.

By *Madam* AGNESI's *Analytical Inſtitutions*,* Vol II, Book III, Sect. I, Article 55, put $u^6 = x$; then we ſhall have

$\frac{6\dot{u}u^5}{(1 - 2u^2 + u^3)^2} = \dot{y}$; and, ſince the denominator of this fraction is the product of
$(1 + u - u^2)^2 \times (1 - u)^2$, we ſhall have by Art. 62 of the ſame Section of the book before mentioned,

$$\frac{6\dot{u}u^5}{(1 - 2u^2 + u^3)^2} = \left\{ \begin{array}{l} \frac{36\dot{u} + 102\dot{u}u + 48\dot{u}u^2 - 36\dot{u}u^3}{(1 + u - u^2)^2} \\ + \frac{-36\dot{u} + 42\dot{u}u}{(1-u)^2}. \end{array} \right.$$

Now the fraction $\frac{-36\dot{u} + 42\dot{u}u}{(1-u)^2}$ is evidently equal to theſe two fractions, $\frac{6\dot{u}}{(1-u)^2} - \frac{42\dot{u}}{1-u}$,

the fluents of which are $\frac{6}{1-u} + 42 \times$ h. l. $(1 - u)$.

And, ſince $(1 + u - uu)^2$ is the product of

$\left(\frac{1 + \sqrt{5}}{2} - u\right)^2 \times \left(\frac{-1 + \sqrt{5}}{2} + u\right)^2$, the other fraction on the right-hand ſide of the above equation may be reſolved into four fractions, of which the fluents will be two algebraic quantities and two logarithms.

* An English Translation of this valuable work was made by the late *Profeſſor* COLSON, of *Cambridge*, and just published from the Translator's Manuscript, under the inspection of the Rev JOHN HELLINS, B. D. F. R. S.

Bu

But an easier way of obtaining the fluent of the fraction $\frac{36\dot{u} + 102\dot{u}u + 48\dot{u}u^2 - 36\dot{u}u^3}{(1+u-uu)^2}$ is, to assume it =

$$\frac{A+Bu}{1+u-uu} + C \times \text{h. l.}\,(1+u-uu) + D \times \text{h. l.}\,(\frac{1+\sqrt{5}}{2} - u),$$

and to put the fluxion of this expression = the given fluxion; from which the values of A, B, C, and D will be easily obtained; viz.

$$A = 21\cdot 6, \qquad C = 0\cdot 246315$$
$$B = 34\cdot 8, \qquad D = -36\cdot 492629$$

Now collecting these fluents together we have

$$\begin{cases} \frac{A+Bu}{1+u-uu} + C \times \text{h. l.}\,(1+u-uu) \\ + D \times \text{h. l.}\,(\frac{1+\sqrt{5}}{2} - u) + \frac{6}{1-u} + 42 \times \text{h. l.}\,(1-u), \end{cases}$$

the sum of which quantities, when $u = 0$, is

$A + D \times \text{h. l.}\,(\frac{1+\sqrt{5}}{2}) + 6$. Therefore as x, y, and u vanish together, from the sum of the fluents above found the constant quantities

$A + D \times \text{h. l.}\,(\frac{1+\sqrt{5}}{2}) + 6$ must be subtracted and we shall have

$$y = \begin{cases} \frac{A+Bu}{1+u-uu} + C \times \text{h. l.}\,(1-u-uu) \\ + D \times \text{h. l.}\,(\frac{1+\sqrt{5}}{1+\sqrt{5}-2u}) + \frac{6}{1-u} \\ + 42 \times \text{h. l.}\,(1-u) - A - 6; \end{cases}$$

in which expression h. l. $(\frac{1+\sqrt{5}}{1+\sqrt{5}-2u})$ is put for its equal

$-$ h. l. $(\frac{1+\sqrt{5}}{2} - u) +$ h. l. $(\frac{1+\sqrt{5}}{2})$.

And the numerical values of the several terms, when $u = \sqrt{\frac{9}{10}}$ (if my calculation be correct,) are as below; viz.

A +

$$\frac{A+Bu}{1+u-uu}=54{\cdot}85600,\ 42\times \text{h.l.}(1\text{-}u)=-170{\cdot}137$$

$$C\times \text{h.l.}(1+u-uu)=\ 0{\cdot}00418,\quad -A-6=-\ 27{\cdot}6$$

$$-D\times \text{h.l.}\left(\frac{1+\sqrt{5}}{1+\sqrt{5}-2u}\right)=34{\cdot}10775,\qquad \text{The sum}\ -197{\cdot}737$$

$$\frac{6}{1-u}\quad =344{\cdot}69287,$$

The sum $433{\cdot}66080$
$-\ 197{\cdot}73756$

Diff. of the sums $235{\cdot}92324=y$, which was require

The Method of solving this question, as it was printed, by I. I

Using the notation in the proposers solution, (which I h seen) we have

$\frac{6\dot{u}u^5}{(1+2u^2+u^3)^2}=y$, of which fraction the denominator is product of

$(0{\cdot}4533977-0{\cdot}2055694\,u+u^2)^2\times(2{\cdot}2055694+u)^2$; and therefore, (putting $2{\cdot}2055604=a$, for the sake of brevi by Art. 62 of the first section of the book above referred to, may assume

$$\left.\begin{array}{c}\dfrac{A\dot{u}+B\dot{u}u+C\dot{u}u^2+D\dot{u}u^3}{\left(\frac{1}{a}-\frac{u}{aa}+uu\right)^2}\\ +\dfrac{E\dot{u}+F\dot{u}u}{(a+u)^2}\end{array}\right\}=\frac{6\dot{u}u^5}{(1+2u+u^3)^2},$$

from which equation the values of A, B, C, &c. will be h and then, to find the fluent of

$\frac{A\dot{u}+B\dot{u}u+C\dot{u}u^2+D\dot{u}u^3}{\left(\frac{1}{a}-\frac{u}{aa}+uu\right)^2}$, (which is the only difficulty remains), assume it $=$

$$\frac{G+Hu}{\frac{1}{a}-\frac{u}{aa}+uu}+I\times \text{h. l.}\left(\frac{1}{a}-\frac{u}{aa}+uu\right.$$

+

$+ K \times$ circ. arch whose rad. is $\frac{\sqrt{(4a^3-1)}}{2aa}$ and tang. $u - \frac{1}{2aa}$, and put the fluxion of this expression = the given fluxion, by which means G, H, &c. will be determined. (See Art. 40, of the first section of the book before referred to).

The fraction $\frac{E\dot{u} + F\dot{u}u}{(a+u)^2}$ is $= \frac{(E-aF)\dot{u}}{(a+u)^2} + \frac{F\dot{u}}{a+u}$

Now, collecting these fluents together, and correcting them, we have

$$y = \begin{cases} \dfrac{G + Hu}{\frac{1}{a} - \frac{u}{aa} + uu} - aG \\ + I \times \left\{ \text{h.l.}\left(\frac{1}{a} - \frac{u}{aa} + uu\right) + \text{h.l.}\, a \right\} \\ + K \times \left\{ \begin{array}{l} \text{circ. arch of tang. } u - \frac{1}{2aa} \\ + \text{ circ. arch of tang. } \frac{1}{2aa} \end{array} \right\} \text{and radius } \dfrac{\sqrt{(4a^3-1)}}{2aa} \\ + \dfrac{aF - E}{a+u} - F + F \times \left\{ \text{h.l.}\,(a+u) - \text{h.l.}\, a \right\}. \end{cases}$$

N. B. In this problem $\frac{\sqrt{(4a^3-1)}}{2aa}$ is $= \frac{\sqrt{(8aa+3)}}{2a}$.

The same, answered by Mr. Cunliffe.

Put $v^6 = x$, and the equation will become $\frac{6\dot{v}v^5}{(1-2v^2+v^3)^2} = \dot{y}$

Now it is pretty obvious that

$(1-v) \times (1+v-v^2) = 1-2v^2+v^3$: therefore assume

$\frac{Av+B}{(1-v)^2} + \frac{Cv^3+Dv^2+Ev+F}{(1+v-v^2)^2} = \frac{v^5}{(1-2v^2+v^3)^2}$, by reducing the terms to a common denominator, &c. there will be had

$$\left.\begin{array}{l} (A + C - 1)v^5 + \left\{ B + D - 2(A + C) \right\} v^4 \\ + \left\{ E + C - A - 2(B+D) \right\} v^3 + (2A - B + F - 2E + D)v^2 \\ + (A + 2B - 2F + E)\, v + B + F \end{array}\right\} = 0.$$

Making

Making the coefficients of the homologous terms $= 0$, we have, after proper reduction,
$A = 7$, $B = -6$, $C = -6$, $D = 8$, $E = 17$ and $F = 6$.
Wherefore,

$$\frac{Av+B}{(1-v)^2} + \frac{Cv^3+Dv^2+Ev+F}{(1+v-v^2)^2} = \frac{7v-6}{(1-v)^2} + \frac{6+17v+8v^2-6v^3}{(1+v-v^2)^2}.$$

Consequently the fluent of $\frac{v^6\dot{v}}{(1-2v^2+v^3)}$ $=$ $\begin{cases} \text{fluent of } \frac{7v\dot{v}-6\dot{v}}{(1-v)^2} \\ + \text{flu. } \frac{6\dot{v}+17v\dot{v}+8v^2\dot{v}-6v^3\dot{v}}{(1+v-v^2)^2} \end{cases}$

$$= \begin{cases} \frac{1}{1-v} + 7 \times \text{h. l. } (1-v) \\ + \text{flu. } \frac{6\dot{v}+17v\dot{v}+8v^2\dot{v}-6v^3\dot{v}}{(1+v-v^2)^2} \end{cases}$$

Now $\frac{6\dot{v}+17v\dot{v}+8v^2\dot{v}-6v^3\dot{v}}{(1+v-v^2)^2} = \frac{6v\dot{v}-2\dot{v}}{1+v-v^2} + \frac{13v\dot{v}+8\dot{v}}{(1+v-v^2)^2}$ by $\div$.

But the denominators of these fractions will be more simple by taking away the second term; it will also be more obvious to which of the tabular forms the fluents of the several terms belong.

In order to take away the said second term of the denominators, put $z + \frac{1}{2} = v$, then will

$$\frac{6\dot{v}+17v\dot{v}+8v^2\dot{v}+8v^2\dot{v}-6v^3\dot{v}}{(1+v-v^2)^2} = \frac{6v\dot{v}-2\dot{v}}{1+v-v^2} + \frac{13v\dot{v}+8\dot{v}}{(1+v-v^2)^2}$$

$$= \frac{6z\dot{z}}{\frac{5}{4}-z^2} + \frac{\dot{z}}{\frac{5}{4}-z^2} + \frac{13z\dot{z}+14\frac{1}{2}\dot{z}}{(\frac{5}{4}-z^2)^2}$$

$$= \frac{6z\dot{z}}{\frac{5}{4}-z^2} + \frac{13z\dot{z}+14\frac{1}{2}\dot{z}+\dot{z}(\frac{5}{4}-z^2)}{(\frac{5}{4}-z^2)^2}$$

$$= \frac{6z\dot{z}}{\frac{5}{4}-z^2} + \frac{15\frac{3}{4}\dot{z}+13z\dot{z}-z^2\dot{z}}{(\frac{5}{4}-z^2)^2}.$$

The fluent of the first term, viz. $\frac{6z\dot{z}}{\frac{5}{4}-z^2}$ is $-3 \times$ h. l. $(\frac{5}{4}-z^2)$.

Assume $\frac{Mz+N}{\frac{5}{4}-z^2} + Q \times \text{flu.} \frac{\dot{z}}{\frac{5}{4}-z^2} = \text{flu. of} \frac{15\frac{3}{4}\dot{z}+13z\dot{z}-z^2\dot{z}}{(\frac{5}{4}-z^2)^2}$;

which expression put into fluxions is

$$\frac{M\dot{z}(\frac{5}{4}-z^2)+2z\dot{z}(Mz+N)}{(\frac{5}{4}-z^2)^2} + \frac{Q\dot{z}(\frac{5}{4}-z^2)}{(\frac{5}{4}-z^2)^2} = \frac{15\frac{3}{4}\dot{z}+13z\dot{z}-z^2\dot{z}}{(\frac{5}{4}-z^2)^2};$$

dividing by $\frac{\dot{z}}{(\frac{5}{4}-z^2)^2}$, and expanding the result there will

be

be had $(M - Q)z^2 + 2Nz + \frac{5}{4}(M+Q) = 15\frac{3}{4} + 13z - z^2$,
or $(M + 1 - Q)z^2 + (2N - 13)z + \frac{5}{4}(M+Q) - 15\frac{3}{4} = 0$.

Making the co-efficients of the homologous terms $= 0$, there will be had, after proper reduction,

$M = \frac{29}{5}$, $N = \frac{13}{2}$ and $Q = \frac{34}{5}$. Therefore the fluent of

$$\frac{15\frac{3}{4}\dot{z} + 13z\dot{z} - z^2\dot{z}}{(\frac{5}{4} - z^2)^2} = \frac{\frac{29}{5}z + \frac{13}{2}}{\frac{5}{4} - z^2} + \frac{34}{5} \times \text{flu.}\ \frac{\dot{z}}{\frac{5}{4} - z^2}$$

$$= \frac{\frac{29}{5}z + \frac{13}{2}}{\frac{5}{4} - z^2} + \frac{34\sqrt{5}}{25} \times \text{h. l.}\left(\frac{\frac{\sqrt{5}}{2} + z}{\frac{\sqrt{5}}{2} - z}\right),$$

because the fluent of $\frac{\dot{z}}{\frac{5}{4} - z^2} = \frac{1}{\sqrt{5}} \times \text{h. l.}\left(\frac{\frac{\sqrt{5}}{2} + z}{\frac{\sqrt{5}}{2} - z}\right)$.

Consequently the fluent of

$$\frac{6\dot{v} + 17v\dot{v} + 8v^2\dot{v} - 6v^3\dot{v}}{(1 + v - v^2)^2} = \begin{cases} \frac{\frac{29}{5}z + \frac{13}{2}}{\frac{5}{4} - z^2} + \frac{34\sqrt{5}}{25} \times \text{h. l.}\left(\frac{\frac{\sqrt{5}}{2} + z}{\frac{\sqrt{5}}{2} - z}\right) \\ -3 \times \text{h. l.}\left(\frac{5}{4} - z^2\right) \end{cases}$$

$$= \begin{cases} \frac{29v + 18}{5(1 + v - v^2)} + \frac{34\sqrt{5}}{25} \times \text{h.l.}\left(\frac{\frac{\sqrt{5}-1}{2} + v}{\frac{\sqrt{5}+1}{2} - v}\right) \\ -3 \times \text{h. l.}\,(1 + v - v^2), \end{cases}$$

by restoring the value of z.

Therefore the fluent of

$$\frac{v^5\dot{v}}{(1 - 2v^2 + v^3)^2} = \frac{1}{1-v} + 7 \times \text{h.l.}\,(1-v) + \text{flu.}\ \frac{6\dot{v} + 17v\dot{v} + 8v^2\dot{v} - 6v^3\dot{v}}{(1 + v - v^2)^2}$$

 =

$$=\left\{\begin{array}{l} +7\times \text{h.l.}(1-v)+\dfrac{29v+18}{5(1+v-v^2)} \\ +\dfrac{34\sqrt{5}}{25}\times \text{h.l.}\left(\dfrac{\frac{\sqrt{5}-1}{2}+v}{\frac{\sqrt{5}+1}{2}-v}\right)-3\times \text{h.l.}(1+v-v^2). \end{array}\right.$$

But by the queſtion this expreſſion ought to vaniſh when $v=0$; therefore the correct fluent of

$$\left.\begin{array}{l} \dfrac{v^5\dot{v}}{(1-2v^2+v^3)^2} \\ =\dfrac{\dot{y}}{6} \end{array}\right\}=\left\{\begin{array}{l} \dfrac{1}{1-v}+7\times \text{h.l.}(1-v)+\dfrac{29v+18}{5(1+v-v^2)} \\ +\dfrac{34\sqrt{5}}{25}\times \text{h.l.}\dfrac{\frac{\sqrt{5}-1}{2}+v}{\frac{\sqrt{5}+1}{2}-v}+\text{h.l.}\left(\dfrac{\sqrt{5}+1}{\sqrt{5}-1}\right) \\ -3\times \text{h.l.}(1+v-v^2)-\dfrac{23}{5} \end{array}\right.$$

$= 39{\cdot}3204703$, becauſe $v=\sqrt[6]{x}=\sqrt[6]{\frac{9}{10}}=$ $\cdot 982598175$.

Whence $y=6\times(39{\cdot}3204703)=235{\cdot}9228218$.

W. W. R.

J. H. is requeſted to ſend to Mr. Glendinning's *for the Medal for ſolving the Prize Queſtion.*

ARTICLE

ARTICLE XV.

MATHEMATICAL QUESTIONS.

To be anſwered in Number XV.

I. QUESTION 301, *by Mr.* William Francis, *Teacher of the Mathematics, Maidenhead, Berks.*

An octagon of fertile ſoil,
  Whoſe copious bounds were juſt a mile,
Was let unto a baker
    Who, fifty years, did pay no rent;
    What can be due at *five per cent.**
And juſt *two pounds* per acre?

* *Compound Intereſt.*

II. QUESTION 302, *by Maſter* John Golding, *at Mr. Gregory's Academy, Cambridge.*

What is the axis of a globe when the ſuperfices and ſolidity are each expreſſed by the ſame number? 2nd. What is the ſuperficies when the axis and ſolidity are each expreſſed by the ſame number? 3d. What is the ſolidity when the axis and ſuperficies are each expreſſed by the ſame number?

III. QUESTION 303, *by Mr.* Francis.

Required the length of a very thin rectangular plate of braſs, 2 inches wide, that would eaxctly reach from the bottom diagonally acroſs to the top of a cylindrical ale quart, whoſe diameter is 4 inches.

IV. QUESTION 304, *by the late Mr.* John Watts, *of Aubourn, Wilts.*

Two poles were placed upright in the earth at 10 yards diſtance from each other to which a rope was faſtened 5 yards above the ground, but a heavy weight being fixed to the middle thereof depreſſed the rope two yards lower in that point and cauſed the poles to meet at the upper end. Query their length?

V. QUESTION 305, *by Mr.* John Whitley.

Given $\begin{cases} x + y = s\ (2) \\ x^9 + y^9 = b\ (32) \end{cases}$ to find x and y by quadratics?

VI. QUESTION 306, *by Mr.* John Barron, *Spilſby*.

There is an air pump, the capacity of its receiver is to that of the barrel as 9 to 2. Now it is required to determine how many ſtrokes of the piſton will rarify the air in the receiver to the ſame denſity as the atmoſphere, at the altitude of 30 miles, when the denſity at the ſurface is 1?

VII. QUESTION 307, *by Mr.* Francis.

A gentleman having a quadrilateral garden ABCD, right-angled at A, wiſhes to render it more uniform by producing the ſide AD to E, which is alſo to be a right-angle, and ſo ſituated that the $\triangle$ DEC may be equal to the $\triangle$ ADB, formed by the two ſides AB, AD, and the diagonal line BD. Now the front door of his houſe, F, is equally diſtant from B and C, whoſe diſtance is the ſame, or 60 yards, from each other; moreover the lines BD, DC, and DF, are to each other as 9, 12, and 8 reſpectively. Hence the figure and area of the garden are required?

VIII. QUESTION 308, *by* Tyro Philomatheticus.

An elliptical field, the ſum of the diameters of which is 1000 yards, is to be ſold at half a guinea for every yard in length on the tranſverſe. and a guinea for every yard in length on the conjugate; required the dimenſions and area of the field, ſuppoſing it of the moſt advantageous form for the purchaſer?

IX. QUESTION 309, *by Mr.* Francis.

M. Garnerin's balloon appears to be compoſed of a cylinder, whoſe ends are covered with two hemiſpheres. Now its diameter being 24 and length 40 feet, what quantity of ſilk was required to make it? with what weight would it aſcend, admitting it to be filled with air 13 times lighter than atmoſpheric air, and what was the length of the parabolic tract deſcribed in his journey from Lord's ground, ſuppoſing his greateſt elevation to have been 3000 yards, and the diſtance of the places whence he aſcended and where he alighted 10 miles in a direct line?

X.

X. QUESTION 310, *by Mr.* Thornoby.

Required the manner of valuing recurring furds continued *ad infinitum*, or to fhew the manner of finding an equation by which their value may be found? and apply the method to the following

1ft. $\sqrt{a+\frac{b}{\sqrt[n]{a+\frac{b}{\sqrt[n]{a+\frac{b}{\sqrt[n]{a+\frac{b}{\sqrt[n]{a+\&c.}}}}}}}}}$

2nd. $\sqrt{a+\frac{b}{\sqrt[n]{c+\frac{d}{\sqrt[m]{a+\frac{b}{\sqrt[n]{c+\frac{d}{\sqrt[m]{a+\frac{b}{\sqrt[n]{c+\&c.}}}}}}}}}}}$

3rd. $\sqrt{\frac{a}{b+\sqrt[n]{\frac{a}{b+\sqrt[n]{\frac{a}{b+\sqrt[n]{\frac{a}{b+\sqrt[n]{\frac{a}{b+\&c.}}}}}}}}}}$

XI. QUESTION 311, *by Mr.* Gregory, *Cambridge.*

There is a fphere whofe diameter is 12 inches; what is the fide of its infcribed cube; and what is the difference between the folidity of the greateft infcribed octagonal pyramid which can be infcribed in the cube, and that of the greateft circular fpindle which can be infcribed in either of the fix fpaces between the cube and the fphere.

XII. QUESTION 312, *by Mr.* William Peacock.

A Gentleman has a garden bounded by a femicircle FCD, a femi-parabola FCAB, F being the vertex thereof. Now the abfciffa of the parabola, (or diameter of the femicircle,) AB the ordinate (rightly applied) being given, the former 100 the latter 45 yards; it is required to find the length of the long walk CD that can be made in the garden, parallel to AB.

XIII. QUESTION 313, *by Mr.* I. H. Swale.

B is a given point in the circumference of a circle given magnitude and pofition, and P another given point, either wit or without the circle; it is required to draw PAC, meeting the c cle in A and C, fo that if AB, CB be drawn, the fum of th fquares may be equal to a given fpace.

XIV. QUESTION 314, *by Mr.* W. Roceſter.

Having given, as in question 235 (vide No. X.), it is requi to folve the problem when the fum of the fquares of the perp diculars is equal to a given fpace.

XV. QUESTION 315, *by* Limenus, *of Bruton, Somerset.*

Let E be the centre of a circle infcribed in the triangle AB which let meet the bafe A B in a; make Bb = Aa, and E drawn parallel to C b will bifect the bafe in M. Required demonftration?

XVI. QUESTION 316, *by Mr.* William Ticken, *of the Roy Military College, Great Marlow, Bucks.*

At what part of the earth is the twilight abfolutely a minimur on October 12th? and how much is the twilight fhorter the than it is in the latitude 52° 12′ N. on the fame day?

XVII. QUESTION 317, *by Mr.* Nicholas Bottom.

In a given triangle to infcribe the greateft ellipfis poffible.

XVI

XVIII. QUESTION 318, *by* Quid Nunc,

From Bonnycastles Algebra.

To find two whole numbers ſuch, that, if unity be added to either, or to their ſum or difference, the ſums thence ariſing ſhall be all ſquare numbers.

XIX. QUESTION 319, *by Mr.* Boſworth, *Cambridge.*

When the ſun ſhines upon a right cone ſtanding with its baſe upon a horizontal plane, the ſhadow of the cone will be bounded by lines drawn from the vertex of the ſhadow as tangents to the baſe of the cone. Required a demonſtration?

XX. QUESTION 320, *by Mr.* Johnſon, *Birmingham.*

Given the difference of the ſegments of the baſe made by the perpendicular, the ratio which the ſum of the ſquares of the ſides has to the rectangle contained by the perpendicular and a given line, to conſtruct the triangle when the ſum of the baſe and perpendicular is a maximum.

XXI. QUESTION 321, *by Mr.* John Whitley.

If Q be the point in the triangle, from which perpendiculars are drawn to the ſides, ſo that the ſolid contained under them is the greateſt poſſible; three times the ſum of the ſquares of the lines drawn from Q to the angular points of the triangle is equal to the ſum of the ſquares of the ſides of the triangle.

XXII. QUESTION 322, *by Mr.* I. F.

If ABCD be a circle given in magnitude and poſition, and if P be a given point without the ſame, a point Q may be found, within the circle, ſuch, that if any line whatever be drawn through P meeting the circle in C and D, and CQ, DQ be joined, the rectangle CQ × QD ſhall be equal to a certain given ſpace.

XXIII. QUESTION 323, *by Mr.* John Lowry.

Let ABC be a triangle inſcribed in the circle AFCBQ, whoſe centre is O, FDQ a diameter drawn perpendicular to the baſe, and meeting it in D, and let CE be drawn parallel to the baſe AB, and meeting DF in E. I ſay the ſolid contained by the ſquare of the

the difference of the fides, and the line drawn from the vertex to the middle of the bafe, will be a maximum, when the fquare of DF (that part of the diameter above the bafe) is equal to thrice the rectangle DO × EF. Required the demonftration?

XXIV. QUESTION 324, *by* Limenus, *Bruton*.

Join A, B, any two points in a conic fection; bifect AB in C, and draw MN parallel to it, and terminated by the curve; draw MCn and NCm to the fection, and the concourfe of Mm, and Nn will give the point P, from whence PA, PB, being drawn will touch the curve. Required a demonstration?

XXV. QUESTION 325, *by Mr.* John Lowry.

Given the difference of the fegments of the bafe, the excefs of the fquare of one fide above the fpace to which the fquare of the other fide has a given ratio, to conftruct the triangle when the folid contained by the fquare of the perpendicular and the bafe is a maximum.

XXVI. QUESTION 326, *by* Marlovienfis.

Required a fimple finite expreffion for the fum of a flowly converging feries of this form,

$a + bx + cx^2 + dx^3 + ex^4 +$ &c. *ad infinitum*; in which all the terms are pofitive, or added to each other.

It is remarkable that this feries, which admits of perfect fummation, is not to be found in any of the Analytic Writers, *De Moivre*, *Waring*, *Landen*, &c. Nor even in *Clarke's* Summation of Series; though the latter Treatife contains many forms that lead immediately to it.

XXVII. QUESTION 327, *by Mr.* James Cunliffe.

It is required to exhibit the whole area of the curve which is the *locus* of the middle point V, of the chain, independant of feries, from the coincidence of the points A and B, to the time that the chain becomes completely ftretched along the horizontal line AB. See my folution to queftion 265.

XXVIII.

XXVIII. QUESTION 328, *by* Philalethes Cantabrigiensis.

If a ball of lead 10 inches in diameter were suspended at a point by a very slender wire 20 inches in length, and made to vibrate in a vertical plane; and if, at the commencement of the motion, the wire being stretched out tight, should be in a line parallel to the horizon, and the ball in such a position that the wire, if continued, should pass through the centre of it; it is required to determine in what time the wire would first become perpendicular to the horizon?

XXIX. QUESTON 329, *by* Marloviensis.

There is an uniform iron lever ACB, of the second order, the weight of which is 2lb. avoird. *per* inch longitudinally. At C, 26 inches from the fulcrum A, is appended a weight of 100lb. Required what length the lever ought to be, that a power P, applied at the extremity B to raise the weight, may be the least possible?

XXX. PRIZE QUESTION 330, *by* Marloviensis.

In the common cycloid ABC, E and F are two points in the axis BD, equally distant from the middle: from these points lines are drawn parallel to the base, cutting the curve in K and L, The generating circle being described in the situation it would have when it generates the point K, and again when it generates the point L, call the points where these circles touch the base G and H. Now, the axis, and the points E, F, being given, it is required to exhibit, by a concise theorem, the area of the cycloidal space contained within the arcs KG and LH without the aid of fluxions; and also to shew a geometrical method of cutting off a part from the generating circle equal to the same space.

ARTICLE

ARTICLE XVI.

Solution to Colonel Titus's Arithmetical Problem.

By James Ivory, *Esq.*

1. THIS problem requires to find the numbers a, b and c, from the following equations, viz.

$$\left.\begin{array}{r} a^2 + bc = 16 \\ b^2 + ac = 17 \\ c^2 + ab = 18 \end{array}\right\} \ldots \text{(No. 1.)}$$

2. Subtract the first of these equations from the second, and the second from the third, and

$$\left.\begin{array}{r} b^2 - a^2 + ac - bc = 1 \\ c^2 - b^2 + ab - ac = 1 \end{array}\right\} \text{ or } \left\{\begin{array}{l} (b-a) \times (b+a-c) = 1 \\ (c-b) \times (c+b-a) = 1 \end{array}\right.$$

$$\text{therefore } b + a - c = \frac{1}{b-a}$$

$$c + b - a = \frac{1}{c-b}$$

Put now, $b - a = m$, $c - b = n$, and we shall have

$$\left.\begin{array}{r} b - a = m \\ c - b = n \\ b + a - c = \frac{1}{m} \\ c + b - a = \frac{1}{n} \end{array}\right\} \ldots \text{(No. 2.)}$$

3. From the equations (No. 2.) it is easy to get these values of a, b and c; viz.

$$\left.\begin{array}{l} a = n + \frac{1}{m} \\ b = \frac{1}{2}\left(\frac{1}{m} + \frac{1}{n}\right) \\ c = \frac{1}{n} - m \end{array}\right\} \ldots \text{(No. 3.)}$$

In

In order to get an equation that shall contain only m and n and known numbers, I substitute the values of a, b and c, just found, in one of the original equations (No. 1.) as the first; and I get

$$\left(n + \frac{1}{m}\right)^2 + \tfrac{1}{2}\left(\frac{1}{m} + \frac{1}{n}\right) \times \left(\frac{1}{n} - m\right) = 16.$$

We have still to find another equation that shall contain only m and n; but from the equations (No. 2) I get, $c - a = m + n$; and also $2c - 2a = \frac{1}{n} - \frac{1}{m}$: therefore $2m + 2n = \frac{1}{n} - \frac{1}{m}$.

We have thus two equations, which are sufficient to determine m and n; viz.

$$\left.\begin{aligned} \left(n + \frac{1}{m}\right)^2 + \tfrac{1}{2}\left(\frac{1}{m} + \frac{1}{n}\right) \times \left(\frac{1}{n} - m\right) &= 16 \\ 2m + 2n &= \frac{1}{n} - \frac{1}{m} \end{aligned}\right\} \text{(No. 4)}$$

and m and n being obtained from these equations the numbers, a, b, and c that are sought will be had by means of the formulæ (No. 3).

4. It is not very easy (without a good deal of algebraic process) to deduce from the equations (No. 4) a final equation, containing only one unknown quantity: the shortest way of doing this appears to be the following.

$$\left(n + \frac{1}{m}\right)^2 = \left(\frac{mn + 1}{m}\right)^2 = \frac{(mn + 1)^2}{m^2}:$$

Again, $\left(\frac{1}{m} + \frac{1}{n}\right) \cdot \left(\frac{1}{n} - m\right) = \left(\frac{m + n}{mn}\right) \cdot \left(\frac{1 - mn}{n}\right) =$

$\frac{(m + n) \cdot (1 - mn)}{mn^2} = \frac{(m^2 + mn) \cdot (1 - mn)}{m^2n^2}$ (by multiplying both numerator and denominator by m).

Therefore, the first of the equations (No. 4) becomes, by substitution, $\frac{(1 + mn)^2}{m^2} + \frac{(m^2 + mn). (1 - mn)}{2m^2n^2} = 16$; and, putting $mn = x$, $m^2 = y$; $\frac{(1 + x)^2}{y} + \frac{(y + x)(1 - x)}{2x^2} = 16.$

Multiply both sides of the second of the equations (No. 4) by m^2n, and it becomes, $2m^3n + 2m^2n^2 = m^2 - mn$, or $2xy + 2x^2 = - x + y$.

Thus we have these two equations, viz.

$$\left.\begin{aligned}\frac{(1+x)^2}{y}+\frac{(y+x)(1-x)}{2x^2}&=16\\ 2xy+2x^2&=-x+y\end{aligned}\right\}\text{(No 5).}$$

In which $x=mn$, $y=n^2$.

5. It is now easy to deduce a final equation from the two equations (No. 5); because the unknown quantity y is only simply concerned, without any of its powers.

Multiply the first of the two equations by y. Then $(1+x)^2+y\,(y+x).\,\frac{1-x}{2x^2}=16y$: but from the second equation I get $y=\frac{x+2x^2}{1-2x}$; therefore $y+x=\frac{x+2x^2}{1-2x}+x=\frac{2x}{1-2x}$: therefore, by substitution

$$(1+x)^2+\tfrac{1}{2}.\frac{x+2x^2}{1-2x}\times\frac{2x}{1-2x}\times\frac{1-x}{x^2}=16\,\frac{x+2x^2}{1-2x}.$$

which is a final equation and only requires further to be properly reduced to prepare it for resolution.

Omitting the factors common to both numerator and denominator of the second member of the left-hand side, and multiplying all the terms by $(1-2x)^2$ I get

$$(1+x)^2(1-2x)^2+(1+2x)(1-x)=16\,(x+2x^2)(1-2x)$$

or

$$(1-x-2x^2)^2+1+x-2x^2=16x-64x^3$$

or

$$2-x-5x^2+4x^3+4x^4=16x-64x^3$$

or

$$2-17x-5x^2+68x^3+4x^4=0$$

or, multiplying by 4 and transposing

$$8=34.2x+5.4x^2-34.8x^3-16x^4.$$

And putting $2x=z$

$$8=34z+5z^2-34z^3-z^4.$$

6. Having determined z by the preceding equation we shall have $x=\frac{z}{2}$ and $y=\frac{x(1+2x)}{1-2x}=\frac{z}{2}\cdot\frac{1+z}{1-z}$.

Also

Also $mn=x$, $y=m^2$: therefore $m=\sqrt{(\frac{z}{2})}\times\sqrt{(\frac{1+z}{1-z})}$

and $n=\frac{x}{m}=\sqrt{(\frac{z}{2})}\times\sqrt{(\frac{1-z}{1+z})}$: Hence

$$a=n+\frac{1}{m}=\left\{\sqrt{(\frac{z}{2})}+\sqrt{(\frac{2}{z})}\right\}\times\sqrt{(\frac{1-z}{1+z})}=\frac{2-z-z^2}{2}\times\sqrt{(\frac{2}{z-z^3})}$$

$$b=\frac{1}{2}\left\{\frac{1}{m}+\frac{1}{n}\right\}=\frac{1}{2}\sqrt{\frac{2}{z}}\times\left\{\sqrt{(\frac{1+z}{1-z})}+\sqrt{\frac{1-z}{1+z}}\right\}=\sqrt{\frac{2}{z-z^3}}$$

$$c=\frac{1}{n}-m=\left\{\sqrt{\frac{2}{z}}-\sqrt{\frac{z}{2}}\right\}\times\sqrt{(\frac{1+z}{1-z})}=\frac{2+z-z^2}{2}\cdot\sqrt{(\frac{2}{z-z^3})}$$

or $b=\sqrt{(\frac{2}{z-z^3})}$

$$a=\frac{2-z-z^2}{2}\times b.$$

$$c=\frac{2+z-z^2}{2}\times b.$$

The Arithmetical Problem that is resolved above was of some note in its day. It was first published by Dr. Wallis, in his Algebra, and it was proposed to the Doctor by Colonel Silas Titus, a gentleman of the bedchamber to Charles the Second. Colonel Titus told Dr. Wallis, that the author of the problem was the famous Algebraist Dr. John Pell, who had proposed it to the Colonel.

The Solution of the problem given by Dr. Wallis is remarkably operose and inellegant, Mr. Baron Maseres has lately republished Dr. Wallis's solution (in a Vol. of Tracts on the resolution of Algebraic Equations, printed in 1800) and has been at great pains to set forth all the operations and reasonings with his usual clearness and accuracy. In the same vol. of Tracts there is likewise printed another solution of the same problem by Mr. Frend, which in point of simplicity is preferable to Dr. Wallis's solution.

ARTICLE XVII.

Solution of a question (the 16th, proposed at page 69, Vol. I. *but not answered) in this work, by* Mr. JOHN FAREY.

To the Editor of the Mathematical Repository.

SIR,

ON the appearance of the *Gentleman's Diary* in November 1789, I was much ſtruck with the aſſertion of the venerable and profound Mathematician who edited that work, ſupported by a proceſs attempting to prove, that no part of the weight of a cylinder, circumſtanced as in your 16th queſtion Vol. I p. 69, could be ſupported by the ſide of the veſſel; and I ſet about, and produced to my friends, *Mr George Sanderſon*, and others, an inveſtigation of the queſtion (609, *Gent's. Diary*) a copy of which you have below; but ſtill ſuſpecting the concluſions of my proceſs, oppoſed by ſuch high authority; I had recourſe to an experiment, the particulars of which you have alſo below. In a letter ſoon after to the *Rev.* Mr. WILDBORE, I objected to the principle laid down in the Diary for 1790, and recommended an experiment, and having the pleaſure of ſeeing that gentleman in Auguſt 1797, at his houſe in Nottinghamſhire, I re-ſtated my objections, which produced queſtion 753 in the Diary for 1798; accident however prevented my ſending a ſolution to that queſtion as I intended, including a ſolution alſo to queſt. 609. Before cloſing the preſent ſeries of your *Repoſitory* it may not therefore be improper to give a place to what follows, in anſwer to the 16th queſtion therein, which will oblige

Your Obedient Servant,

J. FAREY.

12, *Crown Street, Weſtminſter,*
26th December, 1803.

" On account of the equality of the cylindrical Hoofs, the calculation may without ſenſible error proceed, as though the cylindrical ſtaff was divided perpendicularly to the axis at the point of immerſion: it ſhould alſo be conſidered, that the buoyancy of the fluid will be exerted on the centre of gravity of the immerſed part, in a *vertical* direction, and ſince this forms one of the props or ſupports of the cylinder, the other ſupport i. e. the ſide of the veſſel muſt, to produce an equilibrium, act in a *vertical* direction alſo, and hence it is that the inclination of the ſtaff, or height of the ſide of the veſſel, is not a neceſſary datum, and that if a ſubſtitution be made for it, the ſame will fly off in reducing the equation

tion, and that the preſſure on the ſide, and the quantity or content of the ſtaff immerſed are conſtant quantities, not affected by the inclination of the ſtaff.

Fig. 655, Pl. 32. Let $AB = l$, repreſent the ſtaff, immerſed in the beer to C, and ſupported at A by a perpendicular prop; let D be the middle or centre of gravity of the ſtaff, F the middle or centre of gravity of the part without the beer, and E the middle or centre of gravity of the part immerſed. Put $AC = y$, $m =$ the weight of an inch in length of the ſtaff, and $n =$ the weight of an inch in length of the cylinder of beer diſplaced; then $AB \times m = lm$ is the weight of the ſtaff, $BC \times n$ is the weight of the beer diſplaced $= (l - y) \times n$, which exerts itſelf upwards at its centre of gravity E, and $DE = AC + CE - AD = y + \frac{l-y}{2} - \frac{1}{2}l = \frac{1}{2}y$. Now if a body at D be ſupported by two perpendicular props at A and E; it will be as DE $(\frac{1}{2}y)$: AD $(\frac{1}{2}l)$ :: preſſure at A : preſſure at E, and $(DE \times lm) \div (AD + DE) =$ the preſſure at $A = (\frac{1}{2}y \times lm) \div (\frac{1}{2}l + \frac{1}{2}y) = lmy \div (l + y)$; alſo $(AD \times lm) \div (AD + DE) =$ the preſſure at E, $= (\frac{1}{2}l \times lm) \div (\frac{1}{2}l + \frac{1}{2}y) = l^2m \div (l + y)$, this in the caſe of an equilibrium is equal to the action of the fluid, and $l^2m \div (l + y) = (l - y) \times n$, hence $l^2m = (l^2 - y^2) \times n = l^2n - y^2n$, and $ny^2 = l^2 \times (n - m)$, from whence we obtain $y = l\sqrt{(n-m) \div n}$ the length not immerſed, $m = n \times (l^2 - y^2) \div l^2$ the weight of an inch in length of the ſtaff, $n = m \times l^2 \div (l^2 - y^2)$ the weight of an inch in length of beer diſplaced, $l = y\sqrt{n \div (n - m)}$ the ſtaff's length, and by ſubſtituting the value of y in $lmy \div (l + y)$ we have $lm\sqrt{(n - m) \div n} \div (1 + \sqrt{(n - m) \div n}) =$ the preſſure on the ſide of the veſſel.

Now in the 16th queſtion $l = 36$ inches, $y = 13$; and the diameter of beer diſplaced being ·75 inch, we have ·75 × ·75 × ·7854 = ·4417875 the content of an inch in length, which drawn into the weight of an inch of beer ·5949, gives ·2628194 oz. the weight of an inch in length of beer diſplaced $= n$, and $m = n \times (l^2 - y^2) \div l^2 =$ ·228548 oz. which drawn into 36 the length of the ſtaff, gives $lm =$ 8·227719 oz. the weight of the ſtaff as required. Moreover, $lmy \div (l + y) =$ 2·182864 oz. is the weight ſupported by the ſide of the veſſel, being in this caſe ſomething more than one fourth part of the whole weight of the ſtaff."

Note. Mr. Wildbore makes a ſimilar obſervation to the laſt, at page 2 of the Diary for 1799, but it is proper to remark that the weight there given, is erroneous, becauſe derived on the

 ſup-

supposition of the vessel bearing no part of the weight. Also that $\frac{m}{n} = \cdot 8696$ is the specific gravity of the staff, beer being 1.

"*Experiment.* I procured a cylinder of light wood, which I painted, and when dry found its length 28·12 inches $= l$, its mean diameter 1·7 inches, and its weight 17·65 oz. whence $\frac{17\cdot 65}{28\cdot 12} = \cdot 627667$ oz. $= m$. I fastened a thread to one end thereof, and attached it to one of the scales of a balance, suspended by a pulley over a cistern of water; when the cylinder was immersed and at rest, I put 7·175 oz. into the other scale, which exactly balanced the pull of the thread; after which I lowered and again raised the scales, so that the cylinder passed into all degrees of elevation, and I had the pleasure to observe the scales remain, as nearly as possible in equilibrium, and the thread vertical all the time. I also observed that about 19·5 inches of the cylinder's axis remained above the water at several elevations. When this cylinder was afterwards placed nicely upright in the water, and had sunk to its equilibrium I observed that 13·98 inches were immersed, by which if we divide the cylinder's weight 17·65 we get $n = 1\cdot 262581$, and hence $y = l\sqrt{(n-m) \div n} = 19\cdot 94$ inches, the excess of which above 19·5 inches was I believe in some part occasioned by inaccuracy in measuring that length, but more by the inequalities of the cylinder, for on inverting it, the other end sunk 14·45 inches into the water (instead of 13·98) which will give $n = 1\cdot 221458$, and $y = 19\cdot 61$, much nearer to 19·5 than the above. Further, $lmy \div (l+y) = 7\cdot 323$ oz. which differs but ·148 of an ounce from the actual tension of the thread."

P. S. 1st. In making the above experiment it was curious to observe, that when the cylinder was placed very nicely upright in the water, it sunk till its whole weight was supported by the fluid, the depth immersed being $lm \div n = 13\cdot 98$ inches, when if the cylinder's axis was inclined but ever so little to the vertical line, it suddenly started up out of the water, till only $l - l\sqrt{(n-m) \div n}$ was immersed = about 8·18 inches, and the cylinder would have fell down, had not the thread supported it.

P. S. 2d. Notwithstanding the weight supported by the thread was constant, in all the inclinations of the cylinder tried in the above experiment, yet it is certain, that when the cylinder has become nearly horizontal, the weight on the thread begins at a certain point to decrease, and continues so to do as the cylinder sinks, till at length it floats on the water, and the thread sustains no

part

part of its weight; some of the learned Correspondents to your *New Series of the Repository* will I hope give the investigation of these curious particulars.

J. F.

ARTICLE XVIII.

Demonstrations of Lawsons Propositions Proposed in

ARTICLE LII. VOL. II. P. 465.

PROP. XLVII. Fig. 603. Pl. 31.

Demonstrated by Messrs. Campbell, Dawes, *and* Johnson.

By similar triangles,

AD (CB) : ED : : BF : AB (DC), wherefore
CB·BF : ED·DC : : BF^2 : DC^2, and componendo,
CB·BF + ED·DC : ED·DC : : AF^2 : DC^2; But,
AE : AF : : DE : DC, consequently
EA·AF : ED·DC : : AF^2 . DC^2; therefore
EA·AF = ED·DC + CB·BF. *Q. E. D.*

The same, by Messrs. Lowry, Tyro Philomatheticus, Whalley, *and* Whitley.

Complete the rectangle BCEH, and draw BI perpendicular to AE. Then by similar triangles

AB (DC) : AI : : AE : DE, and
AE : EH (BC) : : BF : FI; therefore,
Rect. CDE = rect. EAI, and rect. CBF = rect. AE·FI; therefore
CDE + CBF = EAI + AE × FI = EAF.
Q. E. D.

PROP. XLVIII. Fig. 604, 605. Pl. 31.

Demonstrated by Messrs. Campbell, Dawes, Tyro Philomatheticus, *and* Whalley. Fig. 604.

Let ABC be any right-angled triangle, and DEBF a rectangle inscribed therein. By similar triangles

DE (BF) : FC : : AE : DF (EB), therefore
BFC : FC^2 : : AEB : EB^2, and perm. et comp.
AEB + BFC : AEB : : DC^2 : EB^2. But
AD : DC : : AE : DF, therefore,
ADC : DC^2 : : AEB : EB^2, or, permutando,

ADC

$$ADC : AEB :: DC^2 : EB^2; \text{ therefore}$$
$$AD \cdot DC = AE \cdot EB + BF \cdot FC.$$

Q. E. D.

The same, by Messrs. Johnson, Lowry, *and* Whitley. Fig. 605.

Let DBEF be a rectangle inscribed in the right angled triangle ABC, and draw FS perpendicular to AC. Then by similar triangles,

$$AD : AE :: ES : EF \ (DB), \text{ and}$$
$$AE : ED \ (BF) :: FC : CS; \text{ therefore}$$
$$\text{Rect. } ADB = AFS, \text{ and rect. } BFC = AE \cdot CS; \text{ wherefore}$$
$$ADB + BFC = AES + AE \cdot CS = AEC.$$

Q. E. D.

PROP. XLIX. Fig. 606, 607, 608. Pl. 31.

Demonstrated by Messrs. Campbell *and* Tyro Philomatheticus. Fig. 606.

Let HF be drawn and produced to meet AB in D, and CT, CG being the two tangents, from the centre O draw OT, OG; join TG, and let TC meet FH in I.

Because the $\angle$s FTH, FGH are right ones, a circle will pass through the points G, F, T, H; and the $\angle$ ITH will be equal to the $\angle$ OTI; therefore the $\angle$ ITH = OTB = OBT = AGT = FHT; consequently the $\angle$ FIT = 2 $\angle$ FHT, and therefore I is the centre of the circle GFTH. In like manner it may be shewn, that GC meets FH in the same point; C and I therefore coincide. Moreover, the $\angle$ FHT = FGT = FBD, and the $\angle$ DFB = TFH; consequently the $\angle$ BDF = FTH, = a right angle.

Q. E. D.

The same, by Messrs. Dawes, Whalley, *and* Whitley. Fig. 607.

Three perpendiculars from the angular points of a $\triangle$ upon the opposite sides, meet in the same point. In the $\triangle$ ABH, AD and BE are perpendicular (Eu. 31, III.); therefore HG passing through F is perpendicular to AB. But it remains to be proved that C is in the line GH; draw EK perpendicular to AB, then the $\triangle$s AEK, AHG being similar, the $\angle$AEK = AHG, but the $\angle$ IEA (CEH) = AEK (Eu. 32, III.); therefore the $\angle$ CEH = CHE, and the $\triangle$ CHE is isosceles; in the same manner the $\triangle$ CHD is proved to be isosceles; and the $\triangle$s HEF, HDF being right angled at E and D, a circle will pass through the points D, F, E, H, whose centre is in the common hypothenuse HF, *but CE = CH* = CD; therefore C is the centre of the circle,

and

and is in the line HG; consequently HC, CF produced will be perpendicular to AB.

Q. E. D.

The same, by Mr. John Lowry. Fig. 608.

Let G, D, be the points where the equal tangents touch the circle, and let the lines joining the points F, H meet the diameter in E; then because GB is perpendicular to AH, and AD to BH, HEF will be perpendicular to AB, by a well known property of plane △s; wherefore the truth of the proposition will appear manifest if we can prove that two equal tangents drawn from the points D, G meet each other in the line HF. Now if they do meet in HF it must be where the radius OQ, drawn perpendicular to GD, meets HF, otherwise they could not be equal.

Let OQ produced meet HF at C and join CD, CG; then because of the right angles at D and E, the points D, B, E, F are in a circle, therefore the ∠ DBG = DEC, but the ∠ DBG at the circumference on the whole arch DG, is = to the ∠ DOC at the centre, on half the arch DQ; therefore the ∠ DEC = DOC, and the points O, E, D, C are in a circle; wherefore the ∠ ODC = OEC = a right angle, therefore CD is a tangent to the circle at D; and in the same manner it may be shewn that CG is a tangent at G.

Since drawing up the above, I have been favoured with the following elegant demonstration of this property by my ingenious and worthy friend Mr. John Fletcher. *I trust he will excuse the liberty I have taken in thus making it public.*

To the centre O draw DO, GO and join GD, DB; then since the angle GCD is the supplement of the angle GOD = the sum of the angles AOG, BOD = twice the sum of the angles ADG, BGD = twice the angle AFG, it follows that the angle GCD is equal to the angle at the centre of a circle passing through G, D, F; and as CG = CD, C is the centre itself, and therefore CF = CD; but the angle DBA is = CDA = CFD = AFE, and as A is common, AEF = ADB = a right angle. *Q. E. D.*

PROP. L. Fig. 609, Pl. 31.

Demonstrated by Messrs. Campbell, Dawes, Johnson, Lowry, Tyro Philomatheticus, Whalley, *and* Whitley.

Let BQ be the chord of 60°, equal to the radius, and join BF, FQ, and EF, then the angles BAF, BQF on the same arch are equal, and the triangles EAF, BFQ are isosceles; therefore they are similar, and AF : EF :: EF (BQ) : FB

Q. E. D.

PROP.

PROP. LI. Fig 610, Pl. 31.

Demonstrated by Messrs. Campbell, Dawes, Johnson, Lowry, Tyro Philomatheticus, Whalley, *and* Whitley.

Let the line AB be divided at C in extreme and mean proportion, that is, such that $AB : AC :: AC : CB$ then it is evident that $AB^2 : AB \times AC :: AB \times AC : AB \times CB$, which is the proportion mentioned in the proposition.

PROP. LII. Fig. 611, Pl. 31.

Demonstrated by Messrs. Campbell, Dawes, Johnson, Lowry, Tyro Philomatheticus, Whalley, *and* Whitley.

Let A, B, C, be the three lines. Then
$A : B :: B : C$, or $A \times C = B^2$, $\therefore A \times (A + C) = A^2 + B^2$.
Consеq. $A^2 + B^2 : A \times B :: A \times (A + C) : A \times B :: A + C : B$.

Q. E. D.

PROP. LIII. Fig. 612, Pl. 31.

Demonstrated by Messrs. Campbell, Dawes, Johnson, Lowry, Tyro Philomatheticus, Whalley, *and* Whitley.

Let ABC be any right angled triangle, and CD perpendicular to the hypothenuse AB.

The triangles ACB, CDB, being equiangular,
$AC : AB :: CD : CB$, or $2AC \times CB = 2AB \times CD$.

Therefore $(AC + CB)^2 = AC^2 + BC^2 + 2AC \times CB$,
$= AB^2 + 2AB \times CD$
$= AB \times (AB + 2CD)$.

Consequently $AB : AC + CB :: AC + CB : AB + 2CD$.

Q. E. D.

PROP. LIV. Fig. 613, 614, 615, 616, 617, 618, Pl. 31.

Demonstrated by Mr. Colin Campbell, *of Liverpool.* Fig. 613.

In AD produced take $DG = AB$, draw GE to meet the circle in K, and join AK, AE, KF. Because
$AB \cdot BG = BE^2$, $AFK = AEK =$ a right $\angle = BAF$ (hyp.) therefore EK is parallel and equal to AD ($= BG$); therefore FB and KG are parallel; wherefore $\angle ABF = AGE = AEB$, and therefore, seeing the $\triangle$s AFB, ABE, BEG, similar,
$AF : AB :: AB : BE :: BE : BG$ or AD.

Q. E. D.

$AB \times BD = EB \times BG = EB \times EH$, and
$AB : EB :: EH : BD$ (HP); but by conftr.
$AB : EB :: EB : BC$, therefore
$EH : HP :: EB : BC$; wherefore CE paffes through P, confequently $\angle BCE = HPE$, and arch $AG = FE$; hence $\angle AEB = FPE$, and $DC = AB$, $DP = AF$, therefore $FB = PC$, and the $\angle ABF = AEB$; the triangles ABF and ABE are therefore fimilar, confequently,
$FA : AB :: AB : BE :: BE : BC$ or AD. *Q. E. D.*

The fame, by Mr. John Whitley, *of Attercliffe Academy.* Fig. 618.

Join DE and through F, draw EFG meeting DA produced in G. Then becaufe the $\angle DAF$ is a right angle, the $\angle DEF$ will alfo be a right angle, therefore
$DB \times BG = BE^2$, but $AD \times AB = BE^2$, therefore
$AB \times AD = DB \times BG$; therefore
$AD : DB :: BG : AB$, and by divifion
$AB : DB :: AG : AB$; and by hypothefis
$AF : AB :: BE : AD$, alfo,
$AB : BE :: BE : AD$, therefore
$AF : AB :: AB : BE$, and
$AF \times BE = AB^2$, therefore
$AG : AF :: BE : DB$; which is true, the triangles AGF, DBE, being evidently fimilar.

ARTICLE XIX.

Demonftrations of Lawfon's Propofitions propofed in

ARTICLE VI. VOL. III. P. 71.

PROP. LV. Fig. 612, Pl. 31.

Demonftrated by Meffrs. Campbell, Collins, Dawes, Johnfon, Tyro Philomatheticus, Whalley, *and* Whitley.

Let ACB be a triangle right-angled at C, and CD perpendicular to the hypothenufe AB. Then
$AB : AC :: AC : AD$, and convertendo,
$AB : AB - AC :: AC : AC - AD$, and permutando
$AB : AC :: AB - AC : AC - AD$, wherefore
$AB - AC : AC - AD :: AC : AD$. *Q. E. D.*

PROP.

PROP. LVI. Fig. 619, 620, 621, Pl. 31.

Demonstrated by Messrs. Campbell, Johnson, *and* Whalley. Fig. 619.

Draw the radius OC; then by similar triangles,

KH : HC :: HC : HO, therefore

$KH \cdot HO = HC^2 = BH \cdot HD$; consequently

BH : DH :: BK : KD (Conv. Prop. I.), therefore

FH : HG :: FI : IG (Prop. II.); hence

$FH^2 : FH \cdot FG = HC^2$ :: FI : IG. *Q. E. D.*

The same, by Messrs. Dawes *and* Whitley. Fig. 620.

Upon FG as a diameter describe the semi-circle GLF, join pP, CF, CG; P, p being the centre of the circle and semi-circle; draw the tangent LH, which is equal to CH because their squares are each = to $GH \cdot HF$; and join LG, LF and LI; LI is perpendicular to GF. For the △s HKI and HpP are similar; therefore $PH \cdot HK = pH \cdot HI$; but conceive CP, L$p$ to be joined then $PH \cdot HK = HC^2$ and therefore LI is perpendicular to FG. Now the △s HFC and HCG being similar, as also HFL and HLG.

$HF^2 : HC^2 :: FC^2 : GC^2$, and

$HF^2 : HL^2 (HC^2) :: LF^2 : LG^2$. Therefore

$FL^2 : LG^2 :: FC^2 : CG^2$; but

$FL^2 : LG^2 ::$ FI : IG; therefore

$FC^2 : GC^2 ::$ FI : IG; and therefore

$FH^2 : HC^2 ::$ FI : IG. *Q. E. D.*

The same, by Messrs. Tyro Philomatheticus *and* Collins. Fig. 621.

From the centre O draw OE ⊥ to FG and join OC, then by similar triangles,

HO : HE :: HI : HK therefore

$HE \cdot HI = HO \cdot HK = HC^2 = HG \cdot HF$; whence

HG : HE :: HI : HF, or

2HG : 2HE = HG + HF :: HI = HF + FI : HF, and

2HG — (HG + HF) = FI + IG : FI :: HF + FG : HF;

wherefore by division and conversion

HF : HG :: FI : IG, and therefore

$HF^2 : HG \times HF = HC^2$ :: FI : IG. *Q. E. D.*

PROP. LVII. Fig. 622, 623, 624, 625, Pl. 31.

Demonstrated by Mr. Colin Campbell. Fig. 622.

Draw BG ⊥ and BF ∥ to AC, and O being the centre of the circumscribing circle, join OC.

The angle BAC = to the angle ACB, therefore
ABG = CBG = OCB (becauſe O is in BG), therefore
2OCB = ABC = ACB = CBE + CEB = 2CEB; co ſeq.
GCO = CEB = FBD; alſo FB = AG = OC;
Therefore OC is equal to DB. *Q. E. D.*

The ſame, by Meſſrs. Collins, Dawes, Whalley, *and* Whitley. Fig. 623.

Draw the perpendiculars BI, CK, (which continue to H); their interſection, G, is the centre of the circle circumſcribing the triangle, but AC = CE,

Therefore HC is parallel to DE, but AI = IC, therefore HG = GC, but BG = GC, therefore BGHD is an equilateral parallelogram, and BD = BG. *Q. E. D.*

The ſame, by Mr. Johnſon, *Birmingham.* Fig. 624.

To the centre of the circumſcribing circle O, draw AO, BO, and DO.

Now the ∠s CBE + ABC = a right angle,
therefore the ∠s DBO + DAO = 2 right angles.
Hence, by Eu. 22d. III. a circle will paſs through the points D, B, O, A, therefore in the triangles DBO, BAO,
the ∠ BAO = ABO = BDO = DOB.
But AO = OB, therefore OB = DB. *Q. E. D.*

Corol. The right line DO biſects the ∠ ADB.

The ſame, by Tyro Philomatheticus. Fig. 625.

Deſcribe a circle about the △ and draw BF. Then
∠ DFB = BCA = 60°, and CEB + CBE = BCA;
but CE being equal to CB, ∠ CBE = CEB = 30°,
therefore the complement of CEB = ADE = 60°;
Whence BD = BF, and FAC being a right angle,
FBC = a rt. ∠, but BAC = ABC, ∴ FAB = FBA.

Conſe-

Confequently BF = FA = the fide of a regular hexagon, which, it is well known is equal to the radius of the circumfcribing circle.
Q. E. D.

PROP. LVIII. Fig. 626, Pl. 31.

Demonftrated by Meffrs. Campbell, Collins, Dawes, Johnfon, Tyro Philomatheticus, Whalley, *and* Whitley.

Join CE. Becaufe ∠ ADB = CEB, the triangles ABD, CBE are fimilar, therefore
BE : BD :: CB : AB; but
DC : AD :: CB : AB, (Euc. VI. 3.); confequently
BE : BD :: DC : AD. *Q. E. D.*

PROP. LIX. Fig. 627, Pl. 31.

Demonftrated by Meffrs. Campbell, Collins, Dawes, Johnfon, Tyro Philomatheticus, Whalley, *and* Whitley.

Because BA : BD :: BD : B*e*, and BC : BD :: BD : BE,
the △s ABD, DB*e*; and CBD, DBE, are fimilar;
whence ∠ AED = ∠ ADB, and ∠ D*e*C = ∠ CDB;
and therefore by fimilar triangles
AB : AD :: AD : AE, and CB : CD :: CD : C*e*.
Therefore BAE = AD^2 and BCE = CD^2. *Q. E. D.*

PROP. LX. Fig. 628, Pl. 31.

Demonftrated by Meffrs. Campbell, Collins, Dawes, Johnfon, Tyro Philomatheticus, Whalley, *and* Whitley.

Becaufe the vertical ∠ of the triangle ABC, infcribed in the femi-circle ABC is a right angle, the triangles AFE, EGC are fimilar to ABC, and confequently to one another.
Therefore AE : EF :: GE : EC,
therefore GE·EF = AE·EC = EH^2.
Confequently EG : EH :: EH : EF. *Q. E. D.*

From the preceding demonftration it appears that the triangle need not have been reftricted to an ifofceles one.

ARTICLE XX.

Anſwers to the Mathematical Queſtions propoſed in

ARTICLE V. No. XII.

I. QUESTION 271, *anſwered by* Yanto.

The owners ſtock at the beginning of any year may be eaſily found by means of a table of the different orders of the figurate numbers. Thus at the beginning of

the 1ſt year his ſtock will be	1
2d..........................	1
3d..........................	1
4th..........................	1 + 1
5th..........................	1 + 2
6th..........................	1 + 3
7th..........................	1 + 4 + 1
8th..........................	1 + 5 + 3
9th..........................	1 + 6 + 6
10th..........................	1 + 7 + 10 + 1
11th..........................	1 + 8 + 15 + 4
&c..........................	&c. &c. &c.

Hence it appears that the owners ſtock at the beginning of the 31ſt year will be equal to the ſum of

the 31ſt term of the ſeries	1. 1. 1 &c.	=	1
28th......................	1. 2. 3 &c.	=	28
25th......................	1. 3. 6 &c.	=	325
22d......................	1. 4. 10 &c.	=	2024
19th......................	1. 5. 15 &c.	=	7315
16th......................	1. 6. 21 &c.	=	15504
13th......................	1. 7. 28 &c.	=	18564
10th......................	1. 8. 36 &c.	=	11440
7th......................	1. 9. 45 &c.	=	3003
4th......................	1. 10. 55 &c.	=	220.
1ſt......................	1. 11. 66 &c.	=	1
That is, the owners whole ſtock is		=	58425

In like manner all queſtions of this nature ay be very readily anſwered.

Suppoſe

Suppose the number of years, to be 20, instead of 30. Then it is evident that the owners whole stock will be equal to the sum of

the 21st term of the series	1. 1. 1 &c
18th......................	1. 2 3 &c
15th......................	1. 3. 6 &c
12th......................	1. 4. 10 &c
9th......................	1. 5. 15 &c
6th......................	1. 6. 21 &c
3rd......................	1. 7. 28 &c.

The sum in this case will be 1278.

Again, suppose the number of years to be 40 instead of 30, and that the calves begin to breed at the end of every two years.

In this case the owners whole stock at the beginning of the 41st year will be equal to the sum of

the 41st term of the series	1. 1. 1 &c
39th......................	1. 2. 3 &c
37th......................	1. 3. 6 &c
35th......................	1. 4. 10 &c
&c.	&c.

The stock in this case is found to be 165580141.

The same, answered by Mr. James Campbell, *at Mr. Flower's Academy, Islington.*

By considering the nature of this question, I observe that the number of cows that calved at the end of these years will be as follows, viz.

	3.	4.	5.	6.	7.	8.	9.	10.	11.	12	....
	1.	1.	1.	2.	3.	4.	6.	9.	13.	19	....
and so on to	27.		28.		29.			30		years.	
	5896.		8641.		12664.		and	18560,		respectively,	

which are found by adding the last to the last but two.

Let $s =$ the whole sum, and w, x, y, and z the four last terms of the series; then

$$1+1+1+2+3+4+6+9+13 \ldots\ldots +w+x+y+z=s, \text{ and}$$

$$1+1+1+2+3+\&c. \ldots\ldots\ldots\ldots +w=s-x-y-z;$$

Take the latter of these from the former, and we have

$$1+1+1+1+2+3+4+6+ \&c. \ldots\ldots\ldots +x=x+y+z;$$

add z to both sides of this equation and we get

$$1+1+1+1+2+3+4+6+ \&c. \ldots\ldots +x+z=x+y+2z,$$

which is evidently the owners whole stock of cows and calves at the end of any number of years, and in the present case is equal to $8641+12664+2\times 18560=58425$. *W. W. R.*

 The

The same, by Mr. William Francis, *Taplow Academy.*

Let the lines in fig. 629. pl. 31. represent the years to the 10th, and the dots thereon the calves produced in their respective years. It will then appear, that the produce of calves, in any year, added to the last year but one, will give the number that will be produced the following year. Hence the number produced at the commencement of each year will be as follows, viz.

1st	2d	3d	4th	5th	6th	7th	8th	9th	10th	11th	12th
1.	0.	0.	1.	1.	1.	2.	3.	4.	6.	9.	13.

13th	14th		29th	30th	31st
19.	28	and so on to	8641.	12664	18560.

The sum of which is 58425 the number required.

Solutions to this question were also received from Messrs. Boole *and* Collins.

II. QUESTION 272, *answered by Mr.* Francis.

Let ABCD (fig. 630, pl. 31.) represent a section of the tub through its axis, the line gB the transverse diameter of the elliptical surface of the water when standing upon the inclined plane.

Draw BE perpendicular and gh parallel to CD, intersecting each other in n. Put CD $= a$, AB $= b$, BE $= h$, and $gh = x$.

Then ED $= \frac{1}{2}(a - b)$, $nh = \frac{1}{2}(x - b)$, and $gn = \frac{1}{2}(x + b)$; and by reason of the parallels ED, nh,

$$\text{ED} : \text{EB} :: nh : n\text{B} = \frac{\text{EB}\cdot nh}{\text{ED}} = \frac{h(x-b)}{a-b}.$$

And by corol. 2d Prob. 26th Sect. 1st of Hutton's Mensuration.

$$\frac{gh\sqrt{(gh\ \text{AB}) - \text{AB}^2}}{gh - \text{AB}} \times {\cdot}2618\,\text{B}n \times \text{AB}, \text{ or}$$

$(x\sqrt{[bx]} - b^2) \times \dfrac{{\cdot}2618\,hb}{a - b}$ will express the content of the ungula gAB, which put $= c$, the cubic inches 2 ale gallons;

Whence $x\sqrt{[bx]} = \dfrac{c(a-b)}{{\cdot}2618hb} + b^2$, and $x = 21{\cdot}4115889$.

And by the principles of trigonometry,

$$\frac{2h(x-b)}{(a-b) \times (x+b)} = {\cdot}170434, \text{ nat. tang. of } \angle\, \text{B}gh,$$

the

the inclination of the plane upon which the tub ſtands. From whence, by help of the beſt tables, the $\angle$ Bgh is found to be 9° 40′½ nearly.

And thus the queſtion was anſwered by Meſſrs. Cunliffe, Marrat, Thornoby, *and* Whitley.

III. QUESTION 273, *anſwered by the Propoſer.*

To avoid complex Fractions, put $12x =$ the diameter of the cone's baſe, 12 being diviſible both by 3 and 4; then, by the rules of menſuration

$$144x^2 \times \frac{3.141593}{4} \times \frac{\text{alt.}}{3} = 2412.743424, \text{ the ſolidity;}$$

whence we find the altitude $= \frac{64}{x^2}$, and conſequently the ſlant height $= \sqrt{(36x^2 + \frac{4096}{x^4})}$; this multiplied into $6x \times 3.141593$ half the circumference of the baſe gives the curve ſurface, that is

$$18.849558x\sqrt{(36x^2 + \frac{4096}{x^4})} = 753.98232.$$ Which equation reduced gives $x = 2$: wherefore the diameter of the baſe is 24, the altitude 16, and the ſlant height 20.

Now with reſpect to a conic ungula formed by a vertical plane; it was firſt ſhewn by Father Guido Grandi, an Italian Geometrician, and has been well demonſtrated by Mr. Emerſon in the Ladies' Diary for 1738*, that the ſegment of the baſe formed by the vertical plane is to the curve ſurface of the ungula, as the radius of the baſe, to the ſlant height of the cone. Find therefore, the verſed ſine of that ſegment of the baſe which contains $\frac{1}{5}$ of it; thus, from a table of circular ſegments the verſed ſine of a ſegment containing $\frac{1}{5}$ of the area of a circle to the diameter 1 is ·254069; therefore

As 1 : 24 :: ·254069 : 6·097656 = AS (fig. 631, pl. 31.).

Then AP — AS = 12 — 6·097656 = 5·902344 = PS;

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* An ingenious inveſtigation of a ſimilar theorem was given by Mr. Cunliffe in No. XIII. *Repoſitory*, page 109. See alſo Dr Hutton's tranſlation of Montucla's Ozanam, Vol. I. page 416.

and

and laftly, as is manifeft from the confideration of Mr. Eme
Theorem, and the nature of fimilar triangles,
As AP (12) : AB (20) :: SP (5·9023 &c) : 9·83724 = VE
diftance required from the vertex of the cone.

Neat folutions to this queftion were alfo received from M Cunliffe, Francis, Thornoby, *and* Whitley.

IV. QUESTION 274, *anfwered by Mr.* Thomas Bazle

The weight p fufpended on the wheel BFCD (fig. 632, pl elevates the weight q fufpended on the axle AEC; it is req to determine the preffure on the axis C.

Through the centre C and the points of contact A and B, DACB. Put CA $=a$, CB $=b$; then it is plain by the pr of the lever, that $\frac{qa}{b}$ is the weight which hung at B would b q at A; and therefore $p - \frac{qa}{b} = \frac{pb - qa}{b}$ is the motive acting at B*. Now the mafs moved reduced to the diftanc or CB is (Sect. 6. Prop. I. Atwood on Motion) $\frac{qa^2 + pb^2}{b^2}$ whence the accelerative force is

$$\frac{pb - qa}{b} \div \frac{qa^2 + pb^2}{b^2} = \frac{b(pb - qa)}{qa^2 + pb^2}.$$

Again, the weight q reduced to the diftance CD = C the Prop. above quoted, is $\frac{qa^2}{b^2}$, and confequently

$\frac{qa^2}{b^2} + \frac{qa}{b}$ is the mafs caufing the tenfion of the ftring pB (

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* The enfuing part of the folution will appear more p cuous, by conceiving (as in fig. 633, pl. 31.) the weight q difperfed (uniformly) in the periphery of the circle AEC, an weight $\frac{qa}{b}$ difperfed in the periphery of the circle BFDC; the weight W $= p - \frac{qa}{b}$ as giving motion to this fyftem.

10. Prop. 1. Ibid.); and therefore that tenſion is

$$\left(\frac{qa^2}{b^2} + \frac{qa}{b}\right) \times \frac{b\,(pb - qa)}{qa^2 + pb^2}$$

that is, the maſs into the accelerating force. Laſtly, add to this, the weight of the ſyſtem, viz. $q + \frac{qa}{b}$, and we have the whole preſſure on the axis

$$= \left(q + \frac{qa}{b}\right) + \left(\frac{qa^2}{b^2} + \frac{qa}{b}\right) \times \frac{b\,(pb - qa)}{qa^2 + pb^2},$$

which is eaſily reduced to $\frac{pq\,(a+b)^2}{qa^2 + pb^2}$.

Corol. If $b = a$, as is the caſe in the pulley we have $\frac{pq \times (2a)^2}{(q+b)a^2} = \frac{4pq}{p+q}$ for the preſſure on the axis.

Hence whatever be the radius of the pulley, the preſſure on the axis is the ſame.

Meſſrs. Marrat *and* Thornoby *alſo ſent ſolutions to this queſtion.*

V. QUESTION 275, *anſwered by Mr.* Cunliffe.

Fig. 634. Pl. 31. From the centre O upon DM let fall the perpendicular On, and join OF. Put $AB = BD = 2a$, $BM = x$, and $MF = MP = y$, then $MO = x - a$ and by Euc. I. 47. $DM = \sqrt{(4a^2 + x^2)}$: and by reaſon of the ſimilar Δs MDB, MOn,

$$Mn = \frac{x\,(x-a)}{\sqrt{(4a^2+x^2)}} \text{ and } On = \frac{2a\,(x-a)}{\sqrt{(4a^2+x^2)}}.$$

$$\text{Again } nF^2 = OF^2 - On^2 = \frac{a^2 \times (8ax - 3x^2)}{4a^2 + x^2};$$

$$\text{Whence } nF = a\sqrt{\left(\frac{8ax - 3x^2}{4a^2 + x^2}\right)}, \text{ and therefore}$$

$$MF = MP = Mn + nF, \text{ that is,}$$

$y = \frac{x(x-a)}{\sqrt{(4a^2 + x^2)}} + a\sqrt{\left(\frac{8ax - 3x^2}{4a^2 + x^2}\right)}$, is an equation expreſſing the nature of the curve which is the locus of the point P. Moreover

$y\dot{x} = \frac{x\dot{x}\,(x-a)}{\sqrt{(4a^2+x^2)}} + a\dot{x}.\sqrt{\left(\frac{8ax - 3x^2}{4a^2 + x^2}\right)}$ is the fluxion of the

the area. The correct fluent of the first member is

$$\frac{x}{2}\sqrt{(4a^2+x^2)}-a\sqrt{(4a^2+x^2)}+2a^2-2a^2\times \text{h.l.}\left(\frac{x+\sqrt{(4a^2+x^2)}}{2a}\right).$$

But the fluent of $a\dot{x}\sqrt{\left(\frac{8ax-3x^2}{4a^2+x^2}\right)}$ is not to be expressed by any method which I am acquainted with, without infinite series; it may be integrated by means of a series as follows. Assume

$$\sqrt{\left(\frac{8ax-3x^2}{4a^2+x^2}\right)} = Ax^{\frac{1}{2}} + Bx^{\frac{3}{2}} + Cx^{\frac{5}{2}} + Dx^{\frac{7}{2}} + \&c.$$

By squaring the equation, and expanding $\frac{8ax-3x^2}{4a^2+x^2}$ by division; and then by equating the homologous terms on each side of the equation, we shall obtain

$$\sqrt{\left(\frac{8ax-3x^2}{4a^2+x^2}\right)} =$$

$$\frac{\sqrt{(2a)}}{2a}\times:\left(2x^{\frac{1}{2}}-\frac{9x^{\frac{3}{2}}}{8a}-\frac{73x^{\frac{5}{2}}}{256a^2}+\frac{165x^{\frac{7}{2}}}{4096a^3}+\frac{18035x^{\frac{9}{2}}}{262144a^4}\ \&c.\right)$$

Therefore the fluent of $a\dot{x}\sqrt{\left(\frac{8ax-3x^2}{4a^2+x^2}\right)} =$

$$\sqrt{(2ax)}\times:\left(\frac{2x}{3}-\frac{3x^2}{40a}-\frac{73x^3}{1792a^2}+\frac{165x^4}{36864a^3}+\frac{18035x^5}{2883584a^4}\ \&c.\right)$$

Wherefore the correct expression for the space DMP, is

$$\frac{x}{2}\sqrt{(4a^2+x^2)}-a\sqrt{(4a^2+x^2)}+2a^2-2a\times \text{h.l.}\left(\frac{x+\sqrt{(4a^2+x^2)}}{2a}\right)$$

$$+\sqrt{2ax}\times:\left(\frac{2x}{3}-\frac{3x^2}{40a}-\frac{73x^3}{1792a^2}+\frac{165x^4}{36864a^3}+\&c.\right.$$

This Question was ingeniously answered by Messrs. Marrat *and* Whitley.

VI. QUESTION 276, *answered by Mr.* George Roy.

Since the pendulum composed of the ball of copper, and a slender rod of given length l in inches, and given weight w ounces, made v vibrations in s seconds, it will be easy to find the distance from the point of suspension to the centre of oscillation of the compound pendulum. For $\frac{v}{s}$ will express the number of vibra-

tions

ons in 1 second; and its reciprocal $\frac{s}{v}$ = the time of one vibra-

on. And as the distances from the points of suspension to the ntres of oscillation of pendulums, are as the squares of the times f vibration,

: have $1^2 : \frac{s^2}{v^2} :: 39\frac{1}{8} : \frac{39\frac{1}{8}s^2}{v^2}$ = PV in fig. 635. pl. 31.

being the centre of oscillation of the compound body. Here nd s being given, PV becomes known, and may, therefore, be noted by L. Now in the figure, PB being $= l$, if C be the tre of gravity of the rod, and O its centre of oscillation, then l PC $= \frac{1}{2} l$ and PO $= \frac{2}{3} l$. Call BI the radius of the ball, then, conceiving it suspended at B, and F its centre of oscil-n, we shall have IF $= \frac{2}{5} x$. And if c = weight in ounces 1 cubic inch of copper, and $n = \cdot 5236 \times 8$; cnx^3, will re-ent the weight of the ball. Now M denoting the mass of the der, and M′ that of the sphere, we have, by the nature of the re of oscillation,

$$PV = \frac{(PO \times PC \times M) + (PF \times PI \times M')}{(PC \times M) + (PI \times M')}.$$

et the respective lines and masses in this expression be denoted ove, and it will become

$$= \frac{(\frac{1}{2} l \times \frac{2}{3} l \times w) + (l + x) \times (l + \frac{7}{5} x) \times cnx^3}{\frac{1}{2} lw + (l + x)\, cnx^3}$$

'his, after a little reduction, becomes

$$\tfrac{7}{5} cnx^5 \left. \begin{matrix} + \frac{12}{5} cln \\ - cLn \end{matrix} \right\} x^4 \left. \begin{matrix} + l^2 cn \\ - Llcn \end{matrix} \right\} x^3 = \tfrac{1}{2} Llw - \tfrac{1}{3} l^2 w:$$

quation which it is not necessary to throw into any other form, l some values of L, l, w, &c. are specified; when x may rea-be found, and consequently cnx^3 the weight of the copper

VII. QUESTION 277, *answered by Mr.* Cunliffe.

Let AB (fig. 636, pl. 31.) represent the cylinder just immersed the fluid, with its axis perpendicular to the surface. By the ciples of hydrostatics the pressure of the fluid upon any part of the surface of the cylinder is equal to the weight of a column the fluid, whose base is equal to the surface of the part, and ght equal to $\frac{1}{2}(Ba)$.

That

That is, the pressure upon the surface of the part Ba, is proportional to the solid content of a prism, whose base is equal to the surface of the part Ba, and height $\frac{1}{2}(Ba)$: But the surface of the part Ba is as the height Ba, therefore the pressure upon the surface of the part Ba is as $(Ba)^2$, or as the square of the height Ba. And consequently the pressure upon the surface of the part Aa, will be as $(BA)^2 - (Ba)^2$. And as the pressures upon these two parts are to be equal by the question, we shall have

$$(BA)^2 - (Ba)^2 = (Ba)^2, \text{ or } (BA)^2 = 2(Ba)^2;$$

wherefore if Ba be taken upon BA equal to the side of a square whose diagonal is BA, a will be the point where the cylinder is to be cut, so that the pressures upon the surfaces of the parts Ba and aA shall be equal to each other.

Et cætris paribus, if it was required to cut the cylinder, so that the pressures upon the surfaces of the two parts might have the given ratio of m to n; that is, that the pressure upon the convex surface of the upper part (Ba) may be to the pressure upon the convex surface of the lower part (aA) as m to n. Then from what has been deduced

$$(Ba)^2 : (BA)^2 - (Ba)^2 :: m : n, \text{ whence}$$

$(Ba)^2 = \dfrac{m\,(BA)^2}{m+n}$ which will determine the point a where the cylinder is to be cut in this case.

The same, answered by Mr. William Marrat, *of Boston.*

Put BA the length of the cylinder $= l$, and let the distance of the section from the surface of the water, or $BA = x$; then, because the pressure at A is as $AB = AC$, the pressure at any point a will be as Ba or ab; therefore the pressure on the part Ba will be as the area of the $\triangle\ abB$, and that against aA as the area $CbaA$; that is, $\frac{1}{2}x^2 = \dfrac{l+x}{2} \times (l-x) = \frac{1}{2}(l^2 - x^2)$, by the question.

Therefore $x = l\sqrt{(\frac{1}{2})}$.

This question was likewise ingeniously answered by Messrs. Francis, Thornoby, *and* Whitley.

VIII. QUESTION 278, *answered by* Ambulator.

After the body had passed over the plane AB, (fig. 670. pl. 32.) if there were no loss of velocity at the angle B, the time of running over the space $BD = 2AB$ with the acquired velocity, would be equal

equal to the time employed in passing down AB; and, in this case, since the times of running down planes of equal altitudes are as their lengths, the problem would be solved by simply setting off from A to BD, AE = 2AB.——But, the whole velocity at B : the velocity along BD :: AB : BC :: aB : Bc :: rad. : cos. B. Hence, since the times are inversely as the velocities, if T be the time of running over BD with the whole velocity acquired in falling down AB, we have cos. B : rad. :: T : $\frac{\text{T}}{\text{cos. B}}$ time of running over BD with the reduced velocity. Or, if AB represent the time down AB, we have cos. B : rad. :: AB : $\frac{\text{AB}}{\text{cos. B}}$ :: AB : $\frac{\text{AB}^2}{\text{BC}}$, for the time along BD. Hence take AE = AB + $\frac{\text{AB}}{\text{cos. B}}$ = AB + $\frac{\text{AB}^2}{\text{CB}}$, and set from A to BD, and AE is the length and position of the required plane.

It was also answered by Mr. Marrat.

IX. QUESTION 279, *answered by Mr.* Cunliffe.

Let QCP (fig. 638. pl. 31.) be the given circle, O its centre; and imagine the bodies A and B to move at the same time from the point P, the body A along the periphery, and the body B along the given chord PQ, in such a manner that the straight line AB connecting them may always pass through the centre O of the circle. It is very obvious that the curve which is the locus of the centre of gravity of the two equal bodies A and B, or which is the same thing the middle point in the straight line connecting them will pass through the points P and Q. Draw the radius OmC perpendicular to PQ, and cutting it in m, and let QGP represent the curve which is the locus of the centre of gravity of the two bodies, and let V be the point where this curve cuts the radius OC, and draw GH perpendicular to OC. Put CV = Vm = c, Om = b, VH = x, and HG = y.

Then mH = $c - x$, and OH = $b + c - x$; And by Euc. I. 47. OG = $\sqrt{(\text{OH}^2 + \text{HG}^2)} = \sqrt{\left\{(b+c-x)^2+y^2\right\}}$

And by reason of the parallels mB, HG;
OH : mH :: OG : BG = GA, and by composition
OH : Om + 2mH :: OG : OG + GA = OA = OC,

whence

whence $(Om + 2mH) \times OG = OH \times OC$, which being expressed algebraically is

$$\left\{b+2(c-x)\right\}\sqrt{\left\{(b+c-x)^2+y^2\right\}} = (b+c-x)\times(b+2c), \text{ or}$$

$$\sqrt{\left\{(b+c-x)^2+y^2\right\}} = \frac{(b+c-x)\times(b+2c)}{b+2(c-x)}, \text{ which}$$

is therefore the equation of the curve which is the locus of the centre of gravity of the two bodies.

When $2(c-x) = -b$, that is $2(x-c) = b$, or $x = c + \frac{1}{2}b$, it is plain that y will be infinite. Hence, it appears that the curve has two infinite legs; and the assymptote is parallel to the given chord QP, and bisects Om.

X. QUESTION 280, *answered by Mr.* Cunliffe.

Let DIB (fig. 639, pl. 31.) be half the harmonical curve, CD its axis, and CB its greatest ordinate. Suppose the rectangle HIKC to be the greatest that can be inscribed in the curve; and with the radius CD and centre C describe a circle cutting the ordinate HI in F, and CB in Q. Put DC $= r$, DH $= x$, HI $= y$, the arc DF $= z$, and $c = 3{\cdot}1416$, half the circumference of a circle whose radius is 1. Then by the property of the curve $y = \frac{az}{r}$, where a is a constant quantity easily determined from the data. Whence HI $\times$ CH $= \frac{az}{r} \times (r-x)$ will express the area of the rectangle CI, which is to be a maximum by the question; therefore $z \times (r-x)$ must be a maximum; putting its fluxion $= 0$, $r\dot{z} - z\dot{x} - x\dot{z} = 0$, whence $z\dot{x} = (r-x)\times\dot{z}$; but it is well known that $\dot{z} = \frac{r\dot{x}}{\sqrt{(2rx-x^2)}}$, by means of which, exterminating $\dot{z}$ out of the preceding equation, and it will become

$$z\dot{x} = \frac{r\dot{x}\,(r-x)}{\sqrt{(2rx-x^2)}}; \text{ whence } z = \frac{r(r-x)}{\sqrt{(2rx-x^2)}};$$

or the arc DF $= \frac{\text{CD}\times\text{CH}}{\text{HF}}$. Now $\frac{\text{CD}\times\text{CH}}{\text{HF}}$ is known to express the tangent of the arc QF, or the cotangent of the arc DF. Wherefore the question is now to find such an arc of a circle as shall be equal to its cotangent. An arc of a circle equal to its cotangent,

tangent, from a few approximations by the method of trial and error will be found to be $49°\ 17'\frac{3}{5}$ very near. The cosine of $49°\ 17'\frac{3}{5}$ to rad. 1 is ·652092, and therefore CH $= \cdot 652092 \times r = 1\cdot 304184$. Also, by the property of the curve and the question $CB = (a \div r) \times$ quadrant $DQ = \frac{1}{2}a \times 3\cdot 1416 = a \times 1\cdot 5708 = 10$; whence $a = 10 \div 1\cdot 5708$. Moreover, the length of the arc

$DF = 1\cdot 7206659$; whence $HI = (a \div r) \times DF = \frac{5}{1\cdot 5708}$

$\times 1\cdot 7206659 = 5\cdot 477037$, and $2HI = 10\cdot 954074$ will be the whole length of the greatest rectangle that can be inscribed in the harmonical curve whose base is 20, and height 2, and from what has been before deduced $CH = 1\cdot 304184$ the height of the rectangle; Hence the area is found to be $14\cdot 286128$.

The same, by Mr. John Whitley, *Attercliffe.*

Let MHQGN (fig. 640, pl. 31.) represent the harmonic curve, and suppose that CHGD is the required rectangle. Put $OI = OF = a = 6\cdot 36618$, $OQ = Om = b = 2$, and $x = OP$. Then by the property of the circle $IF = \sqrt{\left\{(a-x) \times 2a\right\}}$, $EF = 2\sqrt{(a^2 - x^2)}$, and the length of the arc

$$EIF = \frac{8\sqrt{\left\{(a-x) \times 2a\right\}} - 2\sqrt{(a^2-x^2)}}{3} =,$$ by the

property of the curve, to GH. And by sim. Δs,

$OF : OP :: Om : On = \frac{bx}{a}$; therefore

$$GH \times CH = \frac{8\sqrt{\left\{(a-x) \times 2a\right\}} - 2\sqrt{(a^2-x^2)}}{3} \times \frac{bx}{a} = a$$

max. per question; the fluxion of which being made $= 0$, and reduced gives

$(4a - 6x)\ \sqrt{\left\{(a-x) \times 2a\right\}} = a^2 - 2x^2$; from this equation x is found $= 4\cdot 1589$ nearly, and the area of the rectangle $= 14\cdot 268$.

Mr. George Roy *and* Mr. Thornoby *answered it.*

XI. QUESTION 281, *answered by the Proposer, Mr.* Dawes.

In fig. 641, pl. 31. let ASL represent the required spherical triangle, make the ∠ CSL = half the given ∠ ASL, continue SL to G so that SG = 91° 39′ = half the given sum. Draw the great circle GC perpendicular to SG and let it meet SC in C; CL making the ∠ CLG = half the complement of the given ∠ ALS

By Trigonometry

As tang. GLC : tang. GSC :: sin. GS : sin. GL = 51° 41′ 29″.
Hence LS = 39° 57′ 31″, therefore AS = 80° 1′ 26″,
and LA = 63. 19′ 8″, which were required.

XII. QUESTION 282, *answered by* Cap. Geo. Gorry.

It has often been shewn that a whole number, having no integral root, has no terminating decimal root, and it is generally admitted, as stated in the question, that the decimal obtained by extracting any root of a whole number cannot be a repetend; but I know of no investigation of this matter which is so perspicuous, and at the same time so concise, as the one given in Vol. II. of Mr. Manning's Introduction to Arithmetic and Algebra. It may be found at page 84, of that ingenious performance, and is as follows:

"If possible, let N be a number the figures of whose *m*th root circulate; and let $\frac{S}{R}$ be equal to the value of the previous figures, and the value of the circulating part. I say in that case

$$\sqrt[m]{N} = \frac{S}{R}, \text{ or } N = \frac{S^m}{R^m}.$$

For, if N be not equal to $\frac{S^m}{R^m}$, let

$$N = \frac{S^m}{R^m} \pm d = \frac{S^m \pm dR^m}{R^m}; \text{ then } \sqrt[m]{N} = \frac{\sqrt[m]{(S^m \pm dR^m)}}{R};$$

That is, the figures obtained for $\sqrt[m]{N}$ agree throughout with the decimal fraction obtained by dividing the figures of $\sqrt[m]{(S^m \pm dR^m)}$ by R; but the figures of $\sqrt[m]{N}$ by hypothesis agree with those obtained by

by dividing S by R, therefore the figures of $\sqrt[n]{(S^m \pm dR^m)}$ are the figures of S, or $\sqrt[n]{(S^m \pm dR^m)} = S$, which is absurd; therefore in that case $\sqrt[n]{N}$ would be equal to $\frac{S}{R}$; but because, by hypothesis, N has no integral root, neither has it any fractional root; Therefore, the decimal, obtained by the operation of extracting the root of a whole number cannot circulate."

Mr. Cunliffe *sent an answer to this question, and so did Mr.* Whitley.

XIII. QUESTION 283, *answered by Mr.* Marrat.

Analysis. (Fig 642, pl. 31.) Let ABC represent the triangle with a circle inscribed, L being the point of contact thereof with the base AB. Through the centre O draw CO to meet the base in I, also draw CK perpendicular to AB, and let E be the middle of AB. By the question LK is given, and EL is known to be equal to half the difference of the sides AC, BC, which is given by the question. Also by Prop. IV. of Harrison's Propositions in Vol. 1st of the Repository $EI \times EK = EL^2$, whence EI becomes known, that is, the point I is given. Hence the following construction.

In the same right line take $EL = \frac{1}{2}$ the given diff. of the sides, and LK = the diff. of the segs. of the base made by the point of contact of the inscribed circle, and that of those made by the perpend. and draw KC perpend. to EK. Now the point I being determined by the Analysis, from I to KC, apply IC = to the given line bisecting the vertical angle. Draw LO perpend. to EK meeting IC in O; then with the centre O and radius OL describe a circle, and from the point C draw two right lines to touch the circle and meet EK in A and B, and ABC is the triangle required, as is evident from the Analysis.

Messrs. Cunliffe, Dawes, *and* Swale *answered it.*

XIV. QUESTION 284, *answered by* Quinbus Flestrin.

Let CAB (fig. 643, pl. 31.) represent a rectangular quadrant of a given ellipsis, E the point of contact of the inscribed circle with the curve, join CE, and draw ED and EF at right angles to

CA and CB respectively. Also draw GH to touch the curve in E, and meet CA, CB, in G and H respectively. Put $CA = a$, $CB = c$, $CD = EF = x$, and $DE = CF = y$.

By Euc. I. 47. $CE^2 = CD^2 + DE^2 = x^2 + y^2$, and by the known property of the ellipsis $c^2 \times (a^2 - x^2) = a^2y^2$.

Also $CD \times CG = CA^2$, or $CD = CA^2 \div CG = a^2 \div x$;

and $CF \times CH = CB^2$, or $CH = CB^2 \div CF = c^2 \div y$.

Moreover in the right angled triangle GCH, E is the point of contact of the inscribed circle with the hypothenuse GH, therefore by a well known property of the rectangle

$GE \times EH = \frac{1}{2}(CG \times CH) = a^2c^2 \div 2xy$.

And by Lawson's Theorems, Prop. 36, and what has been deduced above

$CD \times CG + CF \times CH = CE^2 + GE \times EH = CA^2 + CB^2$, that is $x^2 + y^2 + a^2c^2 \div 2xy = a^2 + c^2$; from this equation and from the property of the ellipse quoted above, x and y may be determined, and by means of these the radius SE easily found.

The same, by Mr. Passman, *the Proposer.*

Fig. 644, pl. 31. BNKD is the given quadrant of the ellipsis. Let NST represent the greatest inscribed circle, touching the ellipsis in N. Draw NC and OT perpend. to BD; and NM and OS perpend. to LD. AL is a tangent to the ellipsis and circle at the point N. Let $a = BD$, $c = DK$, $x = DC$.

Then, by Emerson's Conic Sections $\frac{a^2}{x} = AD$,

and, by the same $CN = \frac{a}{c}\sqrt{(a^2 - x^2)}$.

Moreover $AC = \frac{a^2}{x} - x = \frac{a^2 - x^2}{x}$, and by Eu. I. 47.

$NA = TA = \sqrt{\left\{\left(\frac{a^2 - x^2}{x}\right)^2 + \frac{c^2}{a^2} \times (a^2 - x^2)\right\}}$; conseq.

$DT = \frac{a^2}{x} - \sqrt{\left\{\left(\frac{a^2 - x^2}{x}\right)^2 + \frac{c^2}{a^2} \times (a^2 - x^2)\right\}}$, the radius of the inscribed circle. By sim. $\triangle$s,

$AC : AN :: CD\ (NM) : NL\ (LS) = x^2\sqrt{\left(\frac{1}{x^2} + \frac{c^2}{a^4 - a^2x^2}\right)}$.

And

And as AC : CN :: AD : DL = $\frac{ac}{\sqrt{(a^2-x^2)}}$.

Hence DS = $\frac{ac}{\sqrt{(a^2-x^2)}} - x^2\sqrt{(\frac{1}{x^2} + \frac{c^4}{a^4-a^2x^2})}$.

But DS is equal to DT, that is,

$$\frac{a^2}{x} - (\frac{a^2-x^2}{x})^2 + \frac{c^2}{a^2} \times (a^2-x^2) =$$

$$\frac{ac}{\sqrt{(a^2-x^2)}} - x^2\sqrt{(\frac{1}{x} + \frac{c^2}{a^4-a^2x^2})}.$$ Or,

$$\frac{a^2}{x}\sqrt{(a^2-x^2)} + (2x^2-a^2)\sqrt{(\frac{a^2-x^2}{x^2} + \frac{c^2}{a^2})} = ac.$$

From this equation x may be found.

The same, by Mr. W. Wallace, of the R. M. College.

Analysis. Let abc (fig. 650, pl. 32.) be the given elliptic quadrant. Suppose the problem resolved, and that the inscribed circle touches the ellipse at d and the axes at e and f. Let ABC be a similar elliptic quadrant, in which there is inscribed a circle DEF of a given magnitude; then, if CA, CB, the semi-axes of this ellipse, be found, it is evident that e, f the points at which the circle touches the axes of the given ellipse may also be found; for, since the circle DEF is given in magnitude, the lines CE, CF are given in magnitude, each being equal to its radius, therefore, if the semi-axes CA, CB be determined, they will be divided at E and F into given segments; but the elliptic quadrants abc, ABC being similar, the semi-axes cb, CB will be similarly divided at f and F, therefore the ratio of cf to fb is given, and cb being given the the point f may be found. In like manner it will appear that the point e may be found; but when the points at which the circle touches the axes are determined, the method of inscribing the circle is obvious. We proceed to investigate the semi-axes AC, BC.

Let G be the centre of the circle EDF. Join DG cutting the axes in H and K, then DHK will be a normal to the curve, therefore by a known proposition in Conics, KH is to HD as the square of the excentricity to the square of the semi-transverse axis, which ratio is given, seeing it is the same as that of the like quantities in the given ellipse abc. In GD take L so that GK may be to GL in the same given ratio of HK to HD; then HK : HD

:: L

:: HG : GL — HD; moreover the point G, and the line CK being both given by position the point L is in a straight line given by position; let this line be NQ.

Take GM = DL, then GL = DM, and GL — HD = DM — HD = HM; hence the proportion HK : HD :: HG : GL — HD becomes HK : HD :: HG : HM; now the ratio of HK to HD being given, as already shewn, the ratio of HG to HM is also given; but the point G and the line CH are both given by position, therefore the point M is in a straight line given by position: Let this line be NR.

We have now found that DG passes through a given point G and meets two lines NQ, NR, both given by position, in L and M, so that DL = GM, hence it appears that the *Locus* of the point D is a hyperbola which passes through G, and has NQ, NR for its asymptotes; but D is also in the circumference of a given circle, therefore the point D, and the position of the normal DG are both determined.

Through D draw a perpendicular to the normal meeting the axes produced in S and T, this line, which is evidently a tangent to the curve at D, will be given by position, therefore the points S, T are both given. Draw DX, DY perpendiculars to the axis, and the points X, Y will also be evidently given. Now, from the nature of the ellipse, the semi-transverse CB is a mean proportional between CT and CY, which lines have been shewn to be both given in a magnitude, therefore the semi-transverse axis is also given in magnitude. In like manner it appears that the semi-conjugate axis CA, which is a mean propotional between CS and CX, is given in magnitude, but the semi-axes AC, BC being once found we have already shewn that the whole of the solution is obvious.

Solutions to this question were also received from Messrs. Geo. Roy, Marrat, *and* Whitley.

XV. QUESTION 285, *answered by Mr.* Swale.

Fig. 646, pl. 31. The base and vertical angle being given, the circumscribing circle from thence becomes known; let this circle be described, in which apply the chord AB equal to the given base of the triangle, draw the diameter FH to bisect AB in E, and join FB. Upon FH produced take FN equal to the given aggregate of the sum of the sides and line bisecting the vertical hen produce NF to R in such a manner that FR × RN

$= FB \times (AB + FB)$; to AB apply $FL = FR$, which produce till it cuts the circle in C, join AC, BC, and ABC is the triangle required

Demonstration. The triangles ACL and FLB are similar, and CL bisects the angle ACB,

therefore $FL : FB :: AB : AC + BC$,

whence $FL \times (AC + BC) = FB \times AB$;

Also $FL : FB :: FB : FC$,

whence $FL \times FC = FB^2$, and by adding these together

$FL \times (AC + BC + FC) = FB \times (AB + FB)$.

But by the constr. $FR \times RN = FL \times RN = FB \times (AB + FB)$.

Consequently $RN = AC + BC + FB$, and by taking away $FR = FL$, there will remain $FN = AC + BC + LC$, the given aggregate of the sum of the sides and line bisecting the vertical angle; therefore the triangle ABC is that required.

Thus it was answered by Messrs. Cunliffe *and* Whitley.

XVI. QUESTION 286, *answered.*

Mr. Gregory, the proposer, informs us that the case exhibited for consideration in this question is the same as one of those described by Dr. Desaguliers, in the Philosophical Transactions, No. 407, or abridgment Vol. VI. p. 307. He also finds that the same case is considered in S. Clark's Mechanics, and some later Treatises on that subject, a circumstance of which he was not aware, at the time he proposed the question.

The solution is deduced from very obvious principles. For the oblique force DP (fig. 534, pl. 28.) is resolvable into two others, one as CD perpendicular to the horizon, and the other equal to CP, and parallel to the horizon; this latter force is also parallel to the beam FC which has no tendency to turn it round its centre of suspension F. The force DC then is the only effective force, and must, because action and re-action are equal, be considered both acting *upwards* at the point P, and downwards at the point D or C. Therefore the effect of the oblique thrust DP will be represented by $(CF \times CD) - (FP \times CD)$, and this effect will be shewn in the motion of the scale D downwards.

Now, rad. : sine $\angle$ DPC :: DP : DC;
that is $1 : \cdot 8660254 :: 8 : 6\cdot 9282032$ stones perpendicular force. And
$(CF - PF) \times CD = \cdot 6 \times 6\cdot 9282032 = 4\cdot 15692192$ stones, additional weight, which must be added to W in order to counterbalance the oblique thrust DP.

Had

But P being beyond C, with respect to F the fulcrum, the an oblique thrust, in DP, would have pushed the arm FC upwards, and this would be counter-balanced by taking away from the weight W, a weight proportional to (FP — FC) × CD.

Mr. Moore also favoured us with a solution to this question.

XVIII. QUESTION 287, *answered by Mr.* Cunliffe.

Let P and r denote the same things as in my solution to Question 277; and n the time required.

Then after the first quarterly payment the debt will be denoted by

$$Pr - 1,$$

after the second by $Pr^2 - r - 2$,

after the third by $Pr^3 - r^2 - 2r - 3$,

after the fourth by $Pr^4 - r^3 - 2r^2 - 3r - 4$;

and therefore after the nth quarterly payment the debt will be denoted by

$$Pr^n - r^{n-1} - 2r^{n-2} - 3r^{n-3} - 4r^{n-4} \text{ \&c.} - n$$

$$= r^n \times \left\{P - \left(\frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} + \frac{4}{r^4} + \text{\&c. } \frac{n}{r^n}\right)\right\}.$$

And by the said solution

$$\frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} + \frac{4}{r^4} + \text{\&c.} + \frac{n}{r^n} = \frac{r^{n+1} + n - (n+1)r}{r^n \times (r-1)^2}$$

Wherefore $r^n \times \left\{P - \left(\frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} + \frac{4}{r^4} + \text{\&c. } \frac{n}{r^n}\right)\right\}$

$$= Pr^n - \frac{r^{n+1} + n - (n+1)r}{(r-1)^2}$$

$$= \frac{Pr^{n+2} - (2P+1) \times r^{n+1} + Pr^n + (n+1) \times r - n}{(r-1)^2}$$

$$= \frac{r^n \times \left\{Pr^2 - (2P+1)r + P\right\} + (n+1) \times r - n}{(r-1)^2}$$

And this expression is to be a maximum by the question,

or $r^n \times \left\{ Pr^2 - (2P+1)r + P \right\} + (n+1) \times r - n$ is to be a maximum; wherefore putting its fluxion $= 0$, and dividing by $\dot{n}$, there will be had

$$\left\{ Pr^{n-2} - (2P+1) \times r^{n+1} + Pr^n \right\} \times \text{h.l.}(r) + r - 1 = 0,$$

which when properly reduced gives

$$r^n = \frac{r-1}{\left\{ r - P \times (r-1)^2 \right\} \times \text{h.l.}(r)} = g,$$

whence again $n = \text{h.l.}(g) \div \text{h.l.}(r) = 12{\cdot}70726$.

XVIII. QUESTION 288, *answered by Mr.* Cunliffe.

Let a denote the diameter of the circle, whose chord is l, and versed sine h; and c the diameter of the circle, whose chord is b, and versed sine h. Also let x denote the perpendicular distance of any section of the groin parallel to the base from the vertex. The sides of the said section will be denoted by $2\sqrt{(ax - x^2)}$ and $2\sqrt{(cx - x^2)}$.

And from the nature of the figure and what is done at Art. 165. Vol. 1st. Simpson's Fluxions,

$$\left. \begin{array}{l} 4\sqrt{(ax - x^2)} \times \dfrac{\frac{1}{2}a\dot{x}}{\sqrt{(ax - x^2)}} \\ + 4\sqrt{(ax - x^2)} \times \dfrac{\frac{1}{2}c\dot{x}}{\sqrt{(cx - x^2)}} \end{array} \right\} = 2a\dot{x} + 2c\dot{x},$$ is the fluxion

of the convex surface; the fluent of which is $2ax + 2cx = 2x(a + c)$, and needs no correction, as the expression vanishes when $x = 0$; writing h for x the expression becomes $2h(a + c)$.

Now by the property of the circle

$$a = (\tfrac{1}{4}l^2 + h^2) \div h = \frac{241}{4} \text{ and } c = (\tfrac{1}{4}b^2 + h^2) \div h = 40;$$

wherefore $2h(a + c) = 802$, the convex surface of the groin.

Again, from what is delivered at Art. 155. Vol. 1. Simpson's Fluxions,

$$4\dot{x}\sqrt{(ax - x^2)} \times \sqrt{(cx - x^2)} = 4x\dot{x}\sqrt{\left\{ ac - (a + c)x + x^2 \right\}}$$

$=$

$=$ the fluxion of the solidity $= \dfrac{4acx\dot{x} - 4(a+c)x^2\dot{x} + 4x^3\dot{x}}{\sqrt{\left\{ac - (a+c)x + x^2\right\}}}$

by throwing the surd into the denominator.

And the fluent thereof when properly corrected is

$$\left\{\frac{(a+c)^2}{2} - \frac{4ac}{3}\right\}\sqrt{(ac)} + \frac{4x^2}{3}\sqrt{\left\{ac-(a+c)x+x^2\right\}},$$
$$-\frac{(a+c)x}{3}\sqrt{\left\{ac-(a+c)x+x^2\right\}}$$
$$-\left\{(a+c)^2-\frac{4ac}{3}\right\}\sqrt{\left\{ac-(a+c)x+x^2\right\}}$$
$$+\left\{ac(a+c)-\frac{(a+c)^3}{4}\right\}\times \text{h. l.}\left\{\frac{\frac{a+c}{2}-x-\sqrt{\left\{ac-(a+c)x+x^2\right\}}}{\frac{a+c}{2}-\sqrt{(ac)}}\right\}$$

and writing h in the place of x, the expression becomes

$$\left\{\frac{(a+c)^2}{2} - \frac{4ac}{3}\right\}\sqrt{(ac)} + \frac{4h^2}{3}\sqrt{\left\{ac-(a+c)h+h^2\right\}}$$
$$-\frac{(a+c)h}{3}\sqrt{\left\{ac-(a+c)h+h^2\right\}}$$
$$-\left\{(a+c)^2-\frac{4ac}{3}\right\}\sqrt{\left\{ac-(a+c)h+h^2\right\}}$$
$$+\left\{ac(a+c)-\frac{(a+c)^3}{3}\right\}\times \text{h.l.}\left\{\frac{\frac{a+c}{2}-h-\sqrt{\left\{ac-(a+c)h+h^2\right\}}}{\frac{a+c}{2}-\sqrt{(ac)}}\right\}$$

And this is the correct expression for the solid content or vacuity of the groin. By a careful calculation, the numerical value of the solidity of the groin from the preceding expression is found to be 1483·688422 cubic inches.

XIX. QUESTION 289, *answered by Mr.* I. T. M'Doneld.

LEMMA. The locus of a point within a given trapezium, whence perpendiculars demitted on the sides are together equal to a given length, is a right line given by position.

DEMON

DEMONS. Let ABCD (fig. 648, pl. 32.) be a given trapezium. At the diftance DE or CH = the given fum of 4 perpendiculars draw EF, GH ∥ to AD, BC, meeting the other fides produced in the points I, F, G, K, and bifect the angles at thofe points by right lines meeting the fides in L, M, N, O; join LM and NO interfecting BC in P and AD in Q; draw PQ and it is the locus. For, through L and M draw RS, TV parallel and equal to DE; LW ⊥ to CD, and MY ⊥ to AB; then becaufe the ∠ K is bifected, LW=LS and LW+LR (= MT +MY) = DE, and LM is the locus of a point, whence perpendiculars demitted on the 3 fides AB, AD, CD are together equal to the fum of the perpendiculars on the 4 fides; which I thus prove: through any point p in LM draw qr ∥ DE, alfo ps ∥ AB, pn ⊥ to AB and po ⊥ to CD, alfo Md ∥ EF;

then LS — MV = LW — MY = Ld, and pr — MV = ph.

But by fim fig. LW : MY :: po : Ms;

dividendo LW : Ld :: po : ph (by fim. △s).

Hence po exceeds Ms as much as pr exceeds MV,

and pn+po=MY+ph=pr; theref. pq+pn + po=qr or DE.

In like manner it may be proved that ON is the locus of a point whence perpendiculars demitted on AB, BC, CD, are together equal to CH or DE. Further, through any point Q′ in PQ, draw ad ∥ AD meeting DE in e; then it is too manifeft to need any demonftration here, that if aBCd were the given trapezium and eE the given fum of the perpendiculars, that Q′ would be the pofition of Q, O′N′ of ON, and that perpendiculars from Q′ on the fides aB, BC, Cd, would together be equal to eE, to which add eD or Qf, and DE is the fum of the perpendiculars on the 4 fides demitted from the point Q′; and the fame may be proved of any other point in the line PQ. *Q. E. D.*

The locus PQ (fig. 649, pl. 32.) of the required point being determined for the fum of the perpendiculars, by the preceding Lemma; next find two right lines RY, RZ (by Prop. XVI. of Stewart's Gen. Theo.) that will be given by pofition fuch that the fum of the fquares of perpendiculars demitted from any point on the fides of the trapezium may be equal to 4 times the fum of the fquares of perpendiculars from the fame point on RY, RZ, together with a given fpace, viz. the fum of the fquares of the perpendiculars Rs, Rt, Rv, Rw (Vide Lowry's Sol. pa. 389, vol. I.) The problem will then be reduced to this: to find a right line given by pofition within a given angle YRZ, fuch that the fum of the fquares of perpendiculars from that point on RY, RZ may be equal to a given fpace, viz. ¼ of the difference between the fum of the fquares Rs, Rt, Rv, Rw, and the fum of the fquares of perpendiculars from the required point on the fides of the tra-

pezium which may be done thus: From the point of intersection e, of PQ and RY, demit $ab \perp$ to RZ, produce YR till $Rc = ab$, draw Pd also $\perp$ to RZ, and $Pe \parallel ac$ meeting $Re \perp$ to ac in e, and produce it till $ef = Pd$; join cf, and take Rg = side of a square equal to the given space; lastly draw $gO \parallel cY$ meeting PQ in O, and O is the point required to be found.

For if Oh, Oi, be made $\perp$ to RY, RZ, it is evident from the construction that $Oi = gk$, and $Oh = Rk$, and $Rk^2 + gk^2 = Oh^2 + Oi^2 = Rg^2$ = given space.

Moreover, by the abovementioned proposition $Rs^2 + Rt^2 + Rv^2 + Rw^2 + 4Oh^2 + 4Oi^2 = On^2 + Op^2 + Oq^2 + Or^2$ = the sum of the squares of the 4 perpendiculars given per question.

SCHOL. The sum of the squares of Rs, Rt, Rv, Rw is a minimum.

XX. QUESTION 290, answered by Mr. John [illegible].

From the data we found the latitude = [illegible] and the sun's declination = [illegible].

When the azimuth circle ZON (fig. 64, pl. 31.) becomes a tangent to the parallel of declination d, d, that point is the position of the sun when a maximum in azimuth. In the right angled triangle POZ there are given PZ = co-latitude and PO = co-declination to find the angle OPZ = 17° 58′ 4″; hence the required time is 10h. 48′ 7″ 44‴.

The same, by Captain Geo. Gorry.

It appears from Art. 137, of Mr. Gregory's Astronomy, that when the sun's declination is equal to the difference between the co-latitude and 18°, then the parallel of declination will just touch the crepusculum circle, and the twilight will just continue all night. Hence 37° 47′ 25″ (co-latitude of Cambridge) — 18 = 19° 47′ 25″ north declination of the sun at the commencement of the period when twilight continues all night, this answers pretty well to May 20th 1803. The latitude of the place where the sun's maximum azimuth from the north is sought, is $\frac{37° 47′ 25″}{2}$ =

$= 18^{\circ}\ 53'\ 42''\frac{1}{2}$. This latitude is therefore lefs than the fun's declination; and the cafe correfponds with what is fhewn at Art. 135, of the work above referred to; where it appears that the azimuth circle ZO will touch the parallel of the fun's declination in the point O and there will be given in the fpherical triangle ZOP, right angled at O, the co-latitude ZP $= 71^{\circ}\ 6'\ 17''\frac{1}{2}$, and the co-declination OP $= 70^{\circ}\ 12'\ 35''$, to find ZPO the hour angle from noon; that is, tang. ZP : tang. OP :: radius : cofine of P $= 17^{\circ}\ 58'\ 4''$; this converted into time gives 1h. 11m. 52f. 16t. time from noon: therefore 10h. 48m. 7f. 44t. is the time in the morning when the azimuth from the north is the greateft, at the given day and place.

The fame, by Tyro Philomathcticus.

It is manifeft that the beginning of the period mentioned in the queftion will be when the fun's declination is $= 37^{\circ}\ 47'\ 25''$ (co-lat. of Cambridge) — $18^{\circ} = 19^{\circ}\ 47'\ 25''$, and the latitude of the place is given $= 18^{\circ}\ 53'\ 42''\frac{1}{2}$; therefore let HZHN be the meridian, H H the horizon, SP the axis, Æ Æ the equator, draw the parallel of declination d, d, and by Prob. XLV. Simpfon's Problems, at the end of his Geometry, defcribe an azimuth circle to touch the parallel dd, and the point of contact O will evidently give the fun's greateft diftance from the north; then an hour circle being drawn through O will fhew the time required. Now, becaufe by fpherics the radius Ad of the parallel dd, is a tangent to the primitive HZP at d, we have $Ad^2 = AP \times AS$, but $Ad = AO$; therefore $AO^2 = AP \times AS$, whence AO is a tangent to the hour circle POS at O; but by Simpfon's Geometry III. 8. Cor. 2. AO produced paffes through the centre B of the azimuth circle ZON and confequently is perpendicular to ZO, and therefore ZOP is a right angle; hence we have, as cotang. lat. : cot. declin. :: radius : cofine of the hour from noon $= 17^{\circ}\ 58'\ 3''\frac{1}{2} =$ 1h. 11m. 52f. 14t. the time required from noon.

It was alfo anfwered by Mr. Marrat.

XXI. QUESTION 291, *anfwered by* Quinbus Fleftrin.

Let ACD (fig. 647, pl. 31.) be a femi-circle, AC, CD, and DB arcs into which its circumference is fubdivided. Draw the chords of thefe arcs and the radii CO, DO. The chords AC,

CD, and DB have obviously the same ratio as their halves; and these half chords are respectively the sines of the angles ½AOC, ½COD, and ½DOB.

And ½AOC + ½COD + ½DOB = 1 right angle.

This being premised, put AB = 2r, AC = 2x, DB = 2y; the ratio of AC to CD [illegible]

AC : CD :: r : b :: 2x = AB : 2b; whence

$2b \times AC$ [illegible] AC

to BD be denoted by that of r to c; viz.

AC : BD :: r : c, whence $r \times BD = c \times AC$, or $ry = cx$.

Then by Eu. I. 47. $BC = \sqrt{(4r^2 - x^2)}$, and $AD = \sqrt{(4r^2 - y^2)}$; And by Emerson's Geometry, Book IV. Prop. 28. $BC \times AD = AB \times CD + AC \times CD$, which being expressed algebraically from what has been deduced is

$\sqrt{(4r^2 - x^2)} \times \sqrt{(4r^2 - y^2)} = 2bx + ry$,

and by squaring the equation

$$\left.\begin{array}{l}(4r^2 - x^2) \times (4r^2 - y^2) \\ \text{or } 16r^4 - 4r^2(x^2 + y^2) + x^2y^2\end{array}\right\} = 4b^2x^2 + 4bx^2y + r^2y^2;$$

or, $bxy + r^2x^2 + r^2y^2 + b^2x^2 = 4r^4$, and exterminating y by means of the equation $ry = cx$ it becomes

$\frac{bcx^3}{r} + (r^2 + b^2 + c^2) \times x^2 = 4r^4$, from whence x may be found.

The same, by Scoticus.

Let φ and ψ denote two of the angles into which the right angle is to be divided; then the sines of the three angles required will be

Cos. $(\varphi + \psi)$, sin. φ, sin. ψ.

So that, putting 1, n, p to denote the given ratios, we have

Cos. $(\varphi + \psi) = n$ sin $\varphi = p$ sin ψ.

But cos. $(\varphi + \psi) =$ cos. φ cos. ψ — sin. φ sin. ψ, therefore substituting for sin. ψ its value $\frac{n}{p}$ sin. φ, and transposing,

$$\text{cos. } \varphi \text{ cos. } \psi = \frac{n}{p} \text{ sin. } \varphi^2 + n \text{ sin. } \varphi.$$

Hence,

Hence, squaring and substituting $1 - \text{sin.}\,\phi^2$ for $\text{cos.}\,\phi^2$, also $1 - \frac{n^2}{p^2}\,\text{sin.}\,\phi^2$ for $\text{cos.}\,\psi^2$ we have this equation

$$(1 - \text{sin.}\,\phi^2)\left(1 - \frac{n^2}{p^2}\,\text{sin.}\,\phi^2\right) = \left(\frac{n}{p}\,\text{sin.}\,\phi^2 + n\,\text{sin.}\,\phi\right)^2$$

which by proper reduction and putting $\text{sin.}\,\phi = x$ becomes

$$\frac{2n^2}{p}x^3 + \left(n^2 + \frac{n^2}{p^2} + 1\right)x^2 - 1 = 0.$$

Or, putting $y = \frac{1}{x}$:

$$y^3 - \left(n^2 + \frac{n^2}{p^2} + 1\right)y - \frac{2n^2}{p} = 0$$

a cubic equation wanting the second term and therefore ready to be resolved by known rules.

Ingenious solutions to this question were also received from Messrs. Cunliffe *and* Whitley.

XXII. QUESTION 292.

Not Answered.

XXIII. QUESTION 293, *answered by Mr.* Bazley, *Bolton.*

It is well known that the sum of n terms of the series

$$x^2 + x^3 + x^4 + \&c. \ldots. x^{n+1} = \frac{x^{n+2} - x^2}{x-1};$$

take the second fluxion and it becomes

$$1.2 + 2.3x + 3.4x^2 + 4.5x^3 + \&c. \ldots. n(n+1)x^{n-1}$$

$= \text{the second fluxion of } \frac{x^{n+2} - x^2}{x-1}$

$$=\left\{\begin{array}{l}\frac{(n+1)(n+2)x^{n}-\ldots(x+x)(x^{n+1}\ldots)+\ldots}{\ldots} \\ -\frac{n\left\{(n+2)x^{n+1}\ldots\right\}}{(x-1)^{2}}\end{array}\right.$$

therefore the sum of n terms of the proposed series is,

$$=x\left\{\begin{array}{l}\frac{(n+1)(n+2)x^{n}-2}{x-1}-\frac{n\left\{(n+2)x^{n+1}-2x\right\}}{(x-1)^{2}} \\ +\frac{2\left(x^{n+2}-x^{2}\right)}{(x-1)^{3}}\end{array}\right.$$

GENERALLY. Let $x^{m}+x^{m+1}+x^{m+2}+$ &c. $x^{m+n-1}=s$,

then $1.2.3$ &c....$m+2.3.4$ &c....$(m+1)x+3.4.5$ &c....$(m+2)x^{2}$ $+4.5.6$ &c....$(m+3)x^{3}+$ &c....$n(n+1)(n+2)$ &c....$(m+n-1)x^{n-1}$ is equal to the mth fluxion of s; supposing $x=1$.

The same, answered by Mr. Cunliffe.

First $x^{2}+x^{3}+x^{4}+x^{5}+$ &c. .... $=\frac{x^{2}}{1-x}$;

and $x^{n+2}+x^{n+3}+x^{n+4}+x^{n+5}+$ &c. .... $=\frac{x^{n+2}}{1-x}$;

The latter series is evidently a continuation of the former after n leading terms; wherefore their difference will be

$x^{2}+x^{3}+x^{4}+x^{5}+$ &c. to n terms $=\frac{x^{2}-x^{n+2}}{1-x}$;

the fluxion of this expression divided by $\dot{x}$ is

$+3x^{2}+4x^{3}+5x^{4}+$ &c. to n terms

$=\frac{(n+1)x^{n+2}-(n+2)x^{n+1}+2x-x^{2}}{(1-x)^{2}}=\frac{v}{(1-x)^{2}}$, by putting

$v=$

$v = (n+1)x^{n+2} - (n+2)x^{n+1} + 2x - x^2$.

Again taking the fluxion of the preceeding expreſſion and dividing by $\frac{\dot{x}}{x}$ there will be had

$1.2x + 2.3x^2 + 3.4x^3 + 4.5x^4 + \&c.$ to n terms

$$= \frac{\frac{x\dot{v}}{\dot{x}} \times (1-x) + 2xv}{(1-x)^3} = \frac{x\dot{v}}{\dot{x}(1-x)^2} + \frac{2xv}{(1-x)^3}.$$

Now by the ſubſtitution

$v = (n+1)x^{n+2} - (n+2)x^{n+1} + 2x - x^2$, taking the fluxion of this equation and dividing by $\frac{\dot{x}}{x}$ gives

$$\frac{x\dot{v}}{\dot{x}} = (n+1)(n+2)x^{n+2} - (n+1)(n+2)x^{n+1} + 2x - 2x^2;$$

wherefore $1.2x + 2.3x^2 + 3.4x^3 + 4.5x^4 + \&c.$ to n terms

$$= \frac{x\dot{v}}{\dot{x}(1-x)^2} + \frac{2xv}{(1-x)^3}$$

$$= \left\{ \begin{array}{l} \frac{(n+1)(n+2)x^{n+2} - (n+1)(n+2)x^{n+1} + 2x - 2x^2}{(1-x)^2} \\ + \frac{2(n+1)x^{n+3} - 2(n+2)x^{n+2} + 4x^2 - 2x^3}{(1-x)^3} \end{array} \right.$$

$$= \left\{ \begin{array}{l} \frac{2x - (n+1)(n+2)x^{n+1}}{(1-x)} \\ + \frac{2(n+1)x^{n+3} - 2(n+2)x^{n+2} + 4x^2 - 2x^3}{(1-x)^3} \end{array} \right.$$

Which is a correct expreſſion for the ſum of n leading terms of the propoſed ſeries.

The same, answered by Tyro Philomatheticus.

This question is a particular case of Prob. XXXI. page 125 Emerson's method of Increments, wherein it is required to find the sum of n terms of the series
$Ma + Na^2 + Oa^3 + Pa^4$ &c. ...Putting v for the coefficient of a in the $(n+1)$th term he gets

$$S = \frac{a^{n+1}v}{a-1} - \frac{a^{n+2}\underset{\cdot}{v}}{(a-1)^2} + \frac{a^{n+3}\underset{\cdot\cdot}{v}}{(a-1)^3} - \frac{a^{n+4}\underset{\therefore}{v}}{(a-1)^4} \text{ \&c.}$$

$\underset{\cdot}{v}$, $\underset{\cdot\cdot}{v}$, and $\underset{\therefore}{v}$ being the 1st, 2nd and 3d increments of v.

In the example before us $a = x$, $v = (n+1)(n+2)$, $\underset{\cdot}{v} = 2(n+2)$, $\underset{\cdot\cdot}{v} = 2$, and $\underset{\therefore}{v} = 0$; therefore the sum of n terms is

$$S = \frac{(n+1)(n+2)x^{n+1}}{x-1} - \frac{2(n+2)x^{n+2}}{(x-1)^2} + \frac{2x^{n+3}}{(x-1)^3};$$

but this requires correction, for when $n = 0$,

then $S = \dfrac{2x}{x-1} - \dfrac{4x^2}{(x-1)^2} + \dfrac{2x^3}{(x-1)^3}$;

whence the sum corrected is $\dfrac{(n+1)(n+2)x^{n+1}}{x-1}$

$$- \frac{2(n+2)x^{n+2}}{(x-1)^2} + \frac{2x^{n+3}}{(x-1)^3} - \frac{2x}{x-1} + \frac{4x^2}{(x-1)^2} - \frac{2x^3}{(x-1)^3},$$

this reduced to a common denominator gives

$$S = \frac{n(n+1)x^{n+3} - 2n(n+2)x^{n+2} + (n+1)(n+2)x^{n+1} - 2x}{(x-1)^3}$$

If x be a fraction the theorem will be better expressed with the signs changed.

This Question was ingeniously answered by Messrs. Limenus, Marrat, Quinbus Flestrin, *and* Thornoby.

XXIV.

XXIV. QUESTION 294, *answered by Mr.* Cunliffe.

ANALYSIS. Let ABCD (fig. 651, pl. 32.) be the trapezium, its diagonals AC, BD interfecting each other in I, and suppose the latter divides the former in the given ratio. Draw Im, Ip, respectively parallel to BC, AB, and terminating in those lines in m and p. Also draw In, Iq, respectively parallel to DC, AD, and terminating in those lines in n and q. Then it is obvious that the side AB is divided at m in the given ratio of the parts of the diagonal AI, IC, and therefore Bm is given, and for the same reason Bp is given, or known, as likewise Dn and Dq.

Then by prop. 36, Lawson's Theorems

$$BI^2 + AI \times IC = BA \times Bm + BC \times Bp$$
$$= \text{a given magnitude; and}$$
$$DI^2 + AI \times IC = DA \times Dn + DC \times Dq$$
$$= \text{a given magnitude.}$$

And the difference of these will give
$DI^2 - BI^2 =$ a given magnitude. Now with the centres q and C, and the radii qI, CB let two circles be described; then the question is evidently reduced to what follows; viz. From the given point D to draw a right line to cut the circles whose radii are qI, and CB in I and B so that $DI^2 - BI^2$ may be of a given magnitude.

Quinbus Flestrin *also answered this question.*

XXV. QUESTION 295, *answered by* Scoticus.

Fig. 675. Pl. 33. Suppose a line drawn through the given point P as required. Through the same point draw any other line given by position, cutting the given lines AL, BM, CN, DO, &c. in a, b, c, d, &c. Then the lines Pa, Pb, Pc, Pd, &c. will be given in magnitude. Join PB, PC, PD, &c. cutting Aa, one of the given lines in β, γ, δ, &c. The lines PB, PC, PD, &c. will be given by position, and therefore the points β, γ, δ, &c. will also be given.

From similar triangles, $Pa : Pb :: P\beta : PB :: \beta L : BM$,

therefore $BM = \frac{Pb}{Pa} \cdot \beta L$, and $BM^2 = \frac{Pb^2}{Pa^2} \cdot \beta L^2$.

In

In like manner $CN^2 = \frac{Pc^2}{Pa^2}.\gamma L^2$, also $DO^2 = \frac{Pd^2}{Pa^2}.\delta L^2$, &

Now by hypothesis

$AL^2 + BM^2 + CN^2 + DO^2 +$ &c. $=$ a given space, therefore also

$AL^2 + \frac{Pb^2}{Pa^2}.\beta L^2 + \frac{Pc^2}{Pa^2}.\gamma L^2 + \frac{Pd^2}{Pa^2}.\delta L^2 +$ &c. $=$ a given spa

But it appears from Dr. Stewart's XV General Theorem, (S Math. Rep. Vol. I. Pa. 349.) that a point X may be found Aa, such, that wherever L be taken

$$\left.\begin{array}{l} AL^2 + \frac{Pb^2}{Pa^2}\beta L^2 + \frac{Pc^2}{Pa^2}.\gamma L^2 \\ + \frac{Pd^2}{Pa^2}.\delta L^2 + \text{\&c.} \end{array}\right\} = \left\{\begin{array}{l} \frac{Pa^2 + Pb^2 + Pc^2 + Pd^2 + \text{\&c.}}{Pa^2}.X \\ + \text{ a given space:} \end{array}\right.$$

Hence it is evident that

$$\frac{Pa^2 + Pb^2 + Pc^2 + Pd^2 + \text{\&c.}}{Pa^2}.XL^2 = \text{a given space.}$$

Therefore XL^2 and consequently XL is given. Thus the co struction of the problem is reduced to the determination of t point X, which may be found as in the place already quoted.

XXVI. QUESTION 296, *answered by Mr.* Cunliffe.

Put $m^2 - n^2 = x$, and $2mn = y$, then
$x^2 + y^2 = (m^2 + n^2)^2$ which is a rational square.

It therefore only remains to find such a relation of m and n shall make

$$x^2 + xy + y^2 = (m^2 + n^2)^2 + 2mn \times (m^2 - n^2)$$
$$= m^4 + 2m^3n + 2m^2n^2 - 2mn^3 + n^4 = \text{a squa}$$

Assume $m^2 + mn + n^2$ for its root, then will

$$m^4 + 2m^3n + 2m^2n^2 - 2mn^3 + n^4 = (m^2 + mn + n^2)^2$$
$$= m^4 + 2m^3n + 3m^2n^2 + 2mn^3 + n^4;$$

Whence $m^2n^2 = -4mn^3$, and $m = -4n$.

But this relation of m and n would give the value of y negative therefore in order to obtain positive values for both x and y, p $m = v - 4n$, then will

$$x^2 + xy + y^2 = (m^2 + n^2)^2 + 2mn \times (m^2 - n^2)$$
$$= v^4 - 14v^3n + 74v^2n^2 - 178vn^3 + 169n^4 = \text{a s}$$

Assum

ſume ($v^2 - 7vn - 13n^2$) for the root of the laſt expreſſion, s, put

$$-14v^3n + 74v^2n^2 - 178vn^3 + 169n^4 = (v^2 - 7vn - 13n^2)^2$$
$$= v^4 - 14v^3n + 23v^2n^2 + 182vn^3 + 169n^4;$$

$$\text{whence } 51v^2n^2 = 360vn^3, \text{ or } v = \frac{120n}{17};$$

$$\text{put } n = 17, \text{ then } v = 120, \text{ and } m = v - 4n = 52;$$
$$\text{whence } x = m^2 - n^2 = (52)^2 - (17)^2 = 2415$$
$$\text{and } y = 2mn = 52 \times 34 = 1768.$$

ſolution to the following queſtion may be eaſily effected from t has been done, viz. To find two fractions of ſuch a kind that their ſum, and ſum of their ſquares may be a rational ſquare; if either of them be added to the ſquare of the other, the ſum be a rational ſquare; and furthermore if their product be racted from either of them, the remainder ſhall be a rational re.

et the two fractions be denoted by $\frac{x}{x+y}$ and $\frac{y}{x+y}$; for their will obviouſly be 1, which is a ſquare number, and the ſum their ſquares will be $\frac{x^2+y^2}{(x+y)^2}$, which is to be a ſquare, there- $x^2 + y^2$ muſt be a ſquare, and if either of them be added to ſquare of the other the ſum will be $\frac{x^2+xy+y^2}{(x+y)^2}$ which is to a ſquare, therefore $x^2 + xy + y^2$ muſt be a ſquare. The two aining conditions will be anſwered by the aſſumption or nota- , viz. if their product be taken from either of them, the re- inder ſhall be a ſquare. The queſtion is therefore reduced to ling ſuch values of x and y as ſhall make $x^2 + y^2$, and $x^2 + xy$ y^2 both rational ſquares, which, from what is before done, will when $x = 2415$, and $y = 1768$, and the required fractions l be $\frac{x}{x+y} = \frac{2415}{4183}$ and $\frac{y}{x+y} = \frac{1768}{4183}$.

Neat ſolutions were received from Limenus *and* Whitley.

XVII. QUESTION 297, *anſwered by Mr.* J. H. Swale.

Fig. 653. Pl. 32. CONSTR. Join PQ and make the △ PQR the given magnitude; through R draw the indefinite right line

RG

RG parallel to PQ. Then through Q, by Prob. 37, Simpson's Geometry, draw the right line QGBC cutting the lines RG, AB, and AC in G, B, and C, so that BC may be equal to QG, and the thing is done.

DEMONS. Draw the lines PG, PB, and PC. Then BC = QG by construction; therefore the △ BPC = △ QPG = △ QRP, the given magnitude. *Q. E. D.*

And thus nearly are the answers given by Messrs. Cunliffe, M'Donald, *and* W. Smith.

XXVIII. QUESTION 298, *answered by Mr.* J. T. M'Donald.

CONSTRUCTION. Fig. 684. Pl. 33. Bisect AP in H, and take HI such that rect. PI·IH = rect. BP·PH; draw ID ‖ to BC meeting AC in D, through which draw PDE.

DEMONSTRATION. Bisect the base AB by CG, and ‖ thereto draw CK; also draw EQ and DF ‖ to AB, and join HL which passes through D. Then because DEF is a maximum, its half DML is also a maximum. If not, through any other point *d* in AC draw H*de*, and *fdg* ‖ and equal to MD, meeting KD produced in *g*; bisect KG in N, and draw N*q* ‖ CG meeting DM in O, *fg* in *o*, LL in *p*, and *ke* in *q*, also draw *ghi* ‖ and equal to DL, and *hr* ‖ and equal to D*g* or O*o*, and *dmn* ‖ and equal also to D*g* meeting DL in *m*. Now by Construction

PI : PB :: HP or HA : HI; dividendo,
PI : DF :: HA : AI, or
PI : DM :: HA : AK, and
HK : KN :: HA : AK; convertendo,
HK : HN :: HA : HK = QL : DM = CL : CM, theref.
HK : HN = CL : CM, and consequently
D*k* : *pq* = *dm* : *dn* or O*o*;
but *dm* is greater than D*k*, and theref. O*o* or *hr* exceeds *pq*.

Then (Euc. I. 35), the parallelograms DM*fg*, DL*eg* are equal, from which if the common triangle D*pg* be taken away and L*pf* added, we have the △ DML equal to *gfi* which exceeds the △ *dfe* as much as *ihg* is less than *ghk*. Further draw any other line H*vy* below D meeting N*q* in *x* and I K in *a*, and draw *vcw* ‖ DM meeting N*q* in *c* and DK in *b*; then

D*a* : *px* :: HK : HN :: HA : HK :: CL : CM,

which, because ∠ C*vy* is greater than CDL is a less ratio than that of C*y* : C*w* or D*a* : O*c*; therefore *px* is greater than O*c*, and the

the space LDvy than MDzw, and DML than vwy. Hence the triangle DML and consequently DEF is a maximum.

Q. E. D.

The same, by Mr. Wm. Smith, *the Proposer.*

LEMMA. 1st. Let ABC (fig. 657, pl. 32.) be any △, P a given point in BA produced, draw any line PDE to cut AC, BC in D and E respectively, and draw EF ∥ AB meeting AC in F. Then if AQ be taken in the right line CA equal to DF and PQ be joined, meeting EL ∥ AC in L; the locus of all the points L will be a parabola whose diameters are ∥ to AC, and the parameter (namely of the diameter bisecting BA) of which is to CH (PH being ∥ BC and meeting CA in H) as (joining PC and drawing EW ∥ AC to meet it in W) FE^2 is to FC·EW.

DEMONSTRATION. By the similar △s PDH, EDG and FAD, EDF the rect. HD·DF = rect. AD·DC therefore (producing EL to meet PH in Y and joining PF meeting EL in O) by parallels rect. YE·EO = KE·EW, that is, since AQ = DF, the rect. YE·KL = rect. KE·EW. Therefore YE·LK : BK·KA :: KE·EW : BK·KA :: FC·EW : FE^2; but per Emerson's Conics, Prop. 25th, Bk. IIId. (P being the parameter, &c.) P·LK = BK·KA; therefore P : YE :: FE^2 : FC·EW. Therefore the locus of L is a parabola, whose parameter is to HC (YE) :: FE^2 : FC·EW.

Q. E. D.

LEMMA 2d. Let P be a given point in the side SA produced of any given △ SAR. Then I say, if PQL be drawn meeting AR, RS in Q and L respectively, and AL be joined, the △ AQL will be a maximum, when (drawing LG ∥ AS meeting AR in G) AQ = RG.

DEMONSTRATION. Draw PZ ∥ AR meeting SR in Z, PV ∥ SR meeting RA in V, and ZX ∥ PA meeting AR in X. Then by similar △s PLZ, PQV, rect. ZL·QV = rect. PZ·PV, and therefore the rect. XG·QV = rect. AV·PZ, that is, wheresoever PQL be drawn the rect. XG·VQ is given. Again the △ ALQ = ½AP·GQ (LK — QA), therefore, when the △ AQL is a max. the line GQ is a max. or the sum of XG and VQ a *minimum*, but the rect. XG·VQ is given, therefore XG + VQ is a *minimum*, when XG = VQ, and when XG = VQ, therefore the line GQ and the △ AQL are both the greatest possible, but by ∥s XR = AV, therefore RG = AQ.

Q. E. D.

Hence the problem may be thus constructed:

Draw any line fe ∥ AB and ew ∥ AC meeting PC in w, determine the line Cd, such that twice Cd may be to CH :: fe^2 : fC·

 ew

ew. Bisect AB in M, and produce PM to K, so that drawing KE ∥ AC, the rect. PK·KM may be equal to Cd·KE (the method of doing which is so simple that it need not be detailed); join PE meeting AC in D, and draw EF ∥ AB, so shall DEF be the maximum △ required.

Demonstration. On AB describe a parabola, as in Lemma 1st, the parameter of whose diameter to the ordinate AB may be equal to twice CE, and its diameter ∥ AC. Produce EK to meet FC, and the parabola in W and L respectively, draw the tangent SLR meeting AB, AH in S and R respectively, draw also LG ∥ AB; join PL (meeting AB in Q) and AL, and through M the middle of AB draw INM*aqgxy* meeting EF, ED, AB, AL, PL, LG the parabola and tangent SR in the points I, N, M, *a*, *q*, *g*, *x*, and *y* respectively. Then by construction, rect. PK·KM = KE·Cd: but PK is to KE as FE : FD. Therefore rect. FE·EI = FD·Cd, that is, by ∥s, Lemma 1st and construction rect. GL·Lg = AQ·Cd; and therefore since GL : AQ :: gL·aq : gL² = aq·Cd, that is, gL² = aq·½P (Parameter) but gL² = gx.P, therefore, aq (= twice gx) = (per nature of the tangent to the parabola) yg, and thence AQ = RG. Therefore, per Lemma 2d, the △ AQL is a maximum within the tangential triangle ARS, and consequently is a max. in the inscribed parabola or curve: but since AQ = DF the △ AQL is always = to △ EDF; therefore the △ EDF is a maximum also. *Q. E. D.*

The same by Mr. John Wright, Jun. *of Norley.*

Analysis. Suppose the thing done, and perpendicular to CB (fig. 658, pl. 32.) draw PK meeting AK drawn parallel to BC in K and BC in I; join CK, and through D, parallel to BC, let a right line be drawn, cutting AB in L, PK in G, and CK in S; demit SM perpendicular to BC, also draw PS cutting BC in V. Then by reason of parallel lines,

GK : KI :: LD : EC, and GK : KI :: GS : IC,

and therefore, by equality of ratios,

GS : IC :: LD : EC, or GS : LD :: IC : EC:

also, because of the sim. △s VMS, PGS; EDF, PLD, it will be

VM : GS :: SM (GI) : PG :: EF : LD; wherefore

VM : EF :: GS : LD :: IC : EC (*supra*) a given ratio, consequently the ratio of VM : EF is given; and because LS is parallel to BC, the triangle VSM has to the triangle EDF the same given ratio of VM to EF; therefore, when the triangle EDF is a maximum, the right angled triangle VSM will be a maximum also.

On

On PK describe the semicircle PYQK, cutting SL and CB produced in Q and Y; draw QI, also PQ cutting IY in R; and complete the parallelogram MSQH. Then, by similar triangles, $\triangle$ PGS : $\triangle$ VSM :: $PG^2 : SM^2\ (QH^2)$

:: $QG^2\ (PG.GK) : HR^2$; that is,

$HR^2 : \triangle$ VSM :: PG.GK : $\frac{1}{2}$PG.GS :: GK : $\frac{1}{2}$GS a given ratio, and therefore when the triangle VSM is a maximum, HR^2 will be a maximum, and consequently HR; but when this is a maximum, the triangle RQI must be so; as is evident from the consideration of the similar triangles PIR and HQR, the side PI being constant. Now, if the tangent TN, terminating in IK and IY produced, be drawn to the circle at Q, it will readily appear, from either of the solutions of Question 595, Vol. III. of Hutton's Diarian Miscellany, that when TH is equal to RI, the triangle RQI is greater than any other which can be formed within the triangle TNI, by lines drawn from P and I, and it must evidently then be greater than any other which can be so formed within the semi-segment IKQY. Having therefore shewn that the triangle RIQ, and of consequence the required one EDF, is a maximum when TH = RI, let QK be joined, and, O being the centre of the semi-circle, draw the radius OQ; then the triangle QGK is manifestly similar to QRH,

and hence $QG{\cdot}RH = GK{\cdot}QH$;

the triangle OGQ is also similar to THQ,

and $OG{\cdot}QH = QG{\cdot}TH\ (QG{\cdot}RI) = QG^2 - QG{\cdot}RH = PG{\cdot}GK - KG{\cdot}QH$;

therefore $PG{\cdot}GK = QH{\cdot}(OG + GK) = QH{\cdot}OK = PG{\cdot}OK - PI{\cdot}OK$;

or $PG{\cdot}OG = PI{\cdot}OK$ = a given magnitude. Whence the following

Construction. Having drawn PN perpendicular to BC in I, to meet AK parallel to BC in K, bisect PK in O, from which point on OK set of OG such, that the rectangle under OG and PG may be equal to the given one $PI{\cdot}IO$; from G, parallel to BC, draw GD meeting AC in D; then, if DF parallel to AB meet BC in F, and PD be drawn cutting BC in E, the triangle EDF will evidently be the maximum required.

XXIX. QUESTION 299, *answered by Mr.* Wallace, *Royal Military College.*

Analysis. Suppose the lines drawn as required. From B, (fig. 673, pl. 32.) one of the given points in the circumference

 of

of the given circle, draw BY to one of the given points in the lines given by position; let BY cut the circle again in C; join AC, and take AD = BY; the points D, C, are evidently both given in position; through these points and the given point Y describe a circle cutting the given line YH in F, join FD cutting AE in K, the point F and therefore the line FK are also given in position. The triangles BHY, AKD, are in all respects equal, for FDCY being a quadrilateral in a circle the outward angle HYB is equal to the inward and opposite angle FDC, or to ADK which is vertical to FDC; the angles CBE, CAE, that is YBH, DAK are also equal, for they are in the same segment of the given circle; therefore the triangles YBH, DAK are equiangular, but they have the sides BY, AK (which are opposite to equal angles in each) equal by construction, therefore the remaining sides of the one triangle are equal to those of the other, so that DK is equal to YH: hence the sum or difference of the squares of XG, YH is equal to the sum or difference of the squares of XG, DK; thus the proposed question is reduced to the following. Let DK, XG be two straight lines given by position and D, X given points in those lines, also let A be a given point without them: It is required to draw a line through A to cut them in K and G so that $DK^2 + XG^2$ may be a given space; or so that $DK^2 - XG^2$ may be equal to a given space. We proceed to consider the question under its new form.

Draw AL parallel to GX cutting KD in L, and draw AM parallel to KD, cutting GX in M, the lines AL, AM are evidently given in magnitude. The triangles KLA, AMG are similar, therefore KL : LA :: AM : MG, hence $KL \times MG = AL \times AM =$ a given space. Draw GO perpendicular GX, and equal to KL, then the given rectangle $KL \times GM$ becomes $GO \times GM$, and since the point M, also the line MG are both given by position, the Locus of O is a given right angled hyperbola having MG for one of its asymptotes; the other will pass through M at right angles to MG. Draw XP perpendicular to XG and equal to DL, thus P will be a given point; draw PQ parallel to XG, meeting OG in Q, then the line PQ is given by position; and since by construction $OG = KL$, also $GQ = PX = DL$, we have $OQ = DK$, now $PQ = GX$, therefore $DK^2 \pm XG^2 = OQ^2 \pm QP^2$.

Now if $YH^2 + GX^2 = DK^2 + XG^2$ be given, then $OQ^2 + QP^2 = OP^2$ is given, therefore the point O is in the circumference of a circle whose centre is a given point P, and radius a given line PO; but the same point is also in a given hyperbola as has been already shewn, therefore the point O is determined, and

hence

hence the position of the line OG and the line GAE, as was required. If again $YH^2 - GX^2 = DK^2 - GX^2$ be given, then $QO^2 - QP^2$ is given so that in this case the LOCUS of O is another right angled hyperbola whose centre is P, and conjugate axis in PQ, and therefore the problem may be resolved by the intersection of two hyperbolas.

In general, let the relation between XG and YH be expressed by any equation whatever involving these indeterminate quantities and known magnitudes only, the same equation will express the nature of a curve which is the locus of the point O. Thus we have shewn how the general problem may be resolved, not only in the two cases mentioned in the question, but in innumerable others which might be proposed. If for example we had $XG + YH =$ a given line then also $PQ + QO =$ a given line, or since PX, XM are given, $MG + GO =$ a given line, but here we may dispense with the hyperbola, for we have only to determine two lines MG, GO whose sum and rectangle are both given; and similar simplifications will apply to other cases. As to the construction it follows so obviously from the analysis that for the sake of brevity it may be omitted.

XXX. OR PRIZE QUESTION 300, *answered by* Limenus, *Bruton, Somerset.*

LEMMA. If the side AB (fig. 659, pl. 32.) of any triangle be divided in P in the duplicate ratio of AC : BC the other two sides, and perpendiculars are let fall from P on these sides, the sum of the squares of those perpendiculars, $PM^2 + PN^2$, is the least possible.

Triangles of the same altitude being as their bases

$\triangle APC : \triangle PCB :: AC \cdot PM : BC \cdot PN :: AP : PB$
$:: \text{(by constr.) } AC^2 : BC^2$; whence

$PM : PN :: AC : BC$, and $\dfrac{PM^2}{AP} = \dfrac{PN^2}{BP}$.

Let P be any point between P and B, and since

$$AB + \frac{Pp^2}{AP} + \frac{Pp^2}{BP} = \left\{\begin{matrix} AP + 2Pp + \dfrac{Pp^2}{AP} \\ +BP - 2Pp + \dfrac{Pp^2}{BP} \end{matrix}\right\} = \frac{Ap^2}{AP} + \frac{Bp^2}{BP}$$

 AB

AB is less than $\frac{Ap^{2}}{AP} + \frac{Bp^{2}}{BP}$;

but because of the similar triangles

APM, A*pm*, and BPN, B*pn*,

$\frac{Ap^2}{AP^2} = \frac{pm^2}{PM^2}$, and $\frac{Bp^2}{BP^2} = \frac{pn^2}{PN^2}$, therefore

$AB = AP + BP$ is also less than $AP \times \frac{pm^2}{PM^2} + BP \times \frac{pn^2}{PN^2}$,

and multiplying the whole by $\frac{PM^2}{AP}$ and its equal $\frac{PN^2}{BP}$,

$PM^2 + PN^2$ is less than $pm^2 + pn^2$:

In like manner the same thing may be proved when p is between A and P.

Fig. 660, pl. 32. Draw ab ∥ to the base AB at any distance therefrom, which divide in R in the duplicate ratio of the sides, and join CR meeting ab in Q; then since

$aQ : bQ :: (AR : BR :: AC^2 : BC^2 ::)\ Ca^2 : Cb^2$, the sum of the squares of QM, QN will be less than of those drawn from any other point in ab, by the preceding lemma.

Now divide either of the sides AC in S in the duplicate ratio of the remaining sides of the triangle, and join BS meeting CR in Q, and join AQ to meet BC in T; then, because (by Simpson's Geom. IV. 23.) BT : TC is in the compound ratio of BR : RA, and AS : SC, it is likewise in the duplicate ratio of BA : AC. Therefore the sums of the squares of the perpendiculars QL, QM, and QL, QN are respectively less than those from any other point in lines drawn through Q ∥ to BC, AC. Whence the point Q being so determined, that the sum of the squares of every two of the three perpendiculars QL, QM, QN drawn from thence to three sides is the least, while the third remains constant, we may assume Q from whence the squares of those perpendiculars make together the least sum.

Moreover the △ ABC being equal to the three AQB, BQC, AQC, its double is $= QL \cdot AB + QM \cdot AC + QN \cdot BC$.

But QM : QN :: (by the demon. to the lem.) $Ca : Cb :: CA : CB$,

therefore the perpendiculars from Q are as the sides on which they stand, and

QM·AB

$QM \cdot AB = QL \cdot AC$, and $QN \cdot AB = QL \cdot BC$, whence
$AB \times 2\triangle ABC = QL \cdot AB^2 + QM \times AB \times AC + QN . AB \cdot BC$
$= QL \times (AB^2 + AC^2 + BC^2)$, and
$QL \times 2\triangle ABC = AB \cdot QL^2 + AC \times QL \times QM + BC \cdot QL \cdot QN$
$= AB \times (QL^2 + QM^2 + QN^2)$; wherefore
$AB : QL :: AB^2 + BC^2 + AC^2 : 2\triangle ABC :: 2\triangle ABC : QL^2 + QM^2 + QN^2$.

Q. E. D.

The same, by Mr. I. T. M'Doneld.

LEMMA. Let there be any number of known quantities AB, AC, AD, &c. (fig. 682, pl. 33.) and as many variable quantities Ax, Ay, Az, &c. such that the sum of their rectangles, $AB \cdot Ax + AC \cdot Ay + AD \cdot Az$, shall be equal to a given space; then, I say, the sum of the squares of the unknown quantities, $Ax^2 + Ay^2 + Az^2$, &c. will be a minimum, when the respective factors have one common ratio to each other, viz. $AB : Ax = AC : Ay = AD : Az$, &c.

Suppose AD Az to be a constant magnitude; and, if the sum of the squares of Ax and Ay be not a min. take Ax' less than Ax, and Ay' greater than Ay in an inverse ratio, that is, $Ax : Ay :: y'y : xx'$, or rect. $Ax . xx' = Ay . y'y$.

Now $Ax^2 - Ax'^2 = Ax x' + Ax'x$, and $Ay'^2 - Ay^2 = Ay'y + Ayy'$; the diff. of these two quantities is theref. $Ax'x \backsim Ay'y$ but rect. $Ay'y$ exceeds rect. $Ax'x$ in as much as the ratio of Ay' to Ax' exceeds that of $x'x$ to $y'y$; and theref. Ay'^2 exceeds Ay^2 more than Ax^2 exceeds Ax'^2; and vice versâ if Ax' be taken greater than Ax, its square would exceed that of Ax more than Ay^2 would exceed Ay'^2. Hence $Ax^2 + Ay^2$ is the least possible when $Ax : AB = Ay : AC$. It is easy to conceive that the proposition is equally true in *more* quantities than two.

This premised, the point Q may be determined as follows: find P (fig. 683, pl. 33.) the centre of gravity of the $\triangle$ ABC, whence demit Pm, Pn, Po, perp. to the sides of the $\triangle$, and find O the centre of a circle passing through the points m, n, o, through which draw PQ, making $OQ = OP$ and Q is the point required, as will appear in the following demonstration of the theorem.

DEMON. Demit the perps. Qp, Qq, Qr, and on the centre O with rad. OP or OQ describe a circle intersecting the perp. from

P

P in a, b, and c. Then, the points m, n, o being equi-distant from O, the rectangles amP, Pnb, ocP, and consequently their equals $Qp.Pm$, $Qq.Pn$, $Qr.Po$ are equal to each other; and as Pm, Pn, Po are inversely as the sides on which they stand, Pm, Pn, Po will be directly as the sides, (according to the Lemma),

for $Qp : Qq = Pn : Pm = AB : BC$;

and $Qq : Qr = Po : Pn = BC : AC$:

$Qp^2 + Qq^2 + Qr^2$ is therefore a minimum.

Again rectangle $Qp.Pm = Qp^2$ — rectangle pQd;

rectangle $Qq \cdot Pn = Qq^2 +$ rectangle nbP,

and rectangle $Qr \cdot Po = Qr^2$ — rectangle ocP;

but as the $\triangle s$ AQB and CQA exceed APB and CPA as much as CQB is less than CPB, so rectangles $pQd + ocP = nbP$, and therefore $Qp.Pm + Qq.Pn + Qr.Po = Qp^2 + Qq^2 + Qr^2$. Now $(AB^2 + BC^2 + AC^2) \times (Qp.Pm + Qq.Pn + Qr.Po)$ and and $(AB \cdot Pm + BC \cdot Pn + AC \cdot Po) \times (AB \cdot Qp + BC \cdot Qq + AC \cdot Po)$ when actually multiplied, will by proper substitution of equal rectangles, be found equal to each other. Hence $AB^2 + BC^2 + AC^2$: twice the area :: twice the area : $Qp^2 + Qq^2 + Qr^2$.

Q. E. D.

The same, by Tyro Philomatheticus.

LEMMA. If in any right line AB, (fig. 661, pl. 32.) a point C be taken, such, that $m.AC^2 + n.CB^2$ is the least possible, then AC is to CB as n to m; m and n being given or constant.

DEMONSTRATION. For let D be any point in AB, then $m.AD^2 + n.DB^2 = m.(AC + CD)^2 + n.(CB - CD)^2 = m.(AC^2 + 2AC \cdot CD + CD^2) + n.(CB^2 - 2CB \cdot CD + CD^2)$,

or, since $AC : CB :: n : m$, $CB = \frac{m}{n} \cdot AC$,

$m.AD^2 + n.DB^2 = m.(AC^2 + 2AC \cdot CD + CD^2) +$

$n \cdot (\frac{m^2}{n^2} \cdot AC^2 - \frac{m}{n} \cdot 2AC \cdot CD + CD^2)$, which is manifestly

greater than $m.AC^2 + n.CB^2$ $(= m.AC^2 + n.\frac{m^2}{n^2} \cdot AC^2)$ by

$\overline{m+n} \cdot CD^2$.

Let

Let ABC (fig. 662, pl. 32.) be the triangle. and QD, QE, QF the perpendiculars mentioned in the queſtion, and CI perpendicular to AB, and ſuppoſe the point Q to be ſomewhere in the line GQH which is drawn parallel to AB; then ſince QD (let it be what it will) is the ſame in every point of GQH, we now have to find QF and QE, ſuch, that $QF^2 + QE^2$ ſhall be a minimum: the line GQH being parallel to AB, the angle QGF = angle BAC, and the angle QHE = angle ABC, and QFG and QFH are right angles, therefore by ſimilar triangles

$$AC : CI :: QG : QF = \frac{CI}{AC}\cdot QG, \text{ and}$$

$$BC : CI :: QH : QE = \frac{CI}{BC}\cdot QH, \text{ whence}$$

$$\frac{CI^2}{AC^2}\cdot QG^2 + \frac{CI^2}{BC^2}\cdot QH^2 = QF^2 + QE^2 = \text{a minimum, and}$$

conſequently by the Lemma ($\frac{CI^2}{AC^2}$ and $\frac{CI^2}{BC^2}$ being given)

$$QG : QH :: \frac{CI^2}{BC^2} \; \frac{CI^2}{AC^2} :: AC^2 : BC^2, \text{ therefore}$$

$$QF : QE \left(:: \frac{CI}{AC}\cdot QG : \frac{CI}{BC}\cdot QH\right) :: \frac{CI}{AC}\cdot AC^2 : \frac{CI}{BC}\cdot BC^2 ::$$

$AC : BC$,

and by proceeding in a ſimilar manner with QE and QD or QF and QD we ſhall get

$QE : QD :: BC : BA$ or $QF : QD :: AC : AB$.

Hence the point Q may be found, by Prob. XXX. Simp. on Conſtr. of Geom. Probs. at the end of his Geom. the perpendiculars (when the ſum of their ſquares is a minimum) being as the ſides on which they fall.

Now twice the area of the triangle is

$AB \times QD + BC \times QE + AC \times QF$, but

$$AB : BC :: QD : QE = \frac{BC}{AB}\cdot QD, \text{ and}$$

$$AB : AC :: QD : QF = \frac{AC}{AB}\cdot QD, \text{ whence}$$

twice

twice the area $= AB \times QD + BC \times \frac{BC}{AB} QD + AC \times \frac{AC}{AB} QD$

$$= \frac{QD}{AB} \times (AB^2 + BC^2 + AC^2),$$

and the ſum of the ſquares of the perpendiculars

$= QD^2 + \frac{BC^2}{AB^2} \cdot QD^2 + \frac{AC^2}{AB^2} \cdot QD^2$

$= \frac{QD^2}{AB^2} \times AB^2 + BC^2 + AC^2)$; hence

$AB^2 + BC^2 + AC^2 : \frac{QD}{AB} (AB^2 + BC^2 + AC^2)$

$:: \frac{QD}{AB} \times (AB^2 + BC^2 + AC^2) : \frac{QD^2}{AB^2} \times (AB^2 + BC^2 + AC^2)$.

Q. E. D.

The ſame, by Mr. W. Smith.

LEMMA. Fig. 663, 664, pl. 32. If the baſe AB of a $\triangle$ be divided in Q ſo that $AQ : QB :: AC^2 : BC^2$ the ſum of the ſquares of the $\perp$s demitted from Q on the ſides will be leſs than if Q were taken any where elſe in the baſe.

DEMONSTRATION. Draw QL ſo that the $\angle ALQ = \angle ABC$; and demit the $\perp$s QR, QP then by conſtruction $AQ : QB :: AQ^2 : AQ \cdot QB$ (by hypotheſis) $:: AC^2 : CB^2$ (by ſim. $\triangle$s ACB, AQL) $:: AQ^2 : QL^2$; therefore $AQ \cdot QB = QL^2$. Now if QT be always erected $\perp$ to AB and equal to QL, the locus of T will be a right line given in poſition, as AT: but by reaſon of the equal $\angle$s L and B in the right angled $\triangle$s QRL, QBP, $QR^2 + QP^2$ will be a minimum when $QL^2 + QB^2$ is ſuch, that is, when $(QT^2 + QB^2)$ BT^2 or BT is ſuch, and it has been proved above that $AQ \cdot QB = (QL^2)$ QT^2; therefore the $\angle$ ATB is right and BT or BT^2 or $BQ^2 + QL^2$ or $QP^2 + QR^2$ is a minimum.

Q. E. D.

Hence it appears that with reſpect to the $\triangle$ ACB the locus of the point Q mentioned in the queſtion is the right line CQ and the

the point itself from which the $\perp$s demitted on the sides have the sum of their squares, a minimum must be a point q such that drawing $MqN \parallel AB$, $MN \cdot Nq$ may be $= BN \cdot NC$, for demitting the $\perp$s qI, qE, qF and drawing $qO \parallel QL$ and $qS \parallel CB$ the $\triangle$s qOF, qNE, qSI are all similar, therefore $qO^2 + qN^2 + qS^2$ is a minimum, that is to say $Nq.NM + NB^2$ is a minimum; now it appears (since $Nq.NM : NC^2$ in a given ratio) that if NG be erected $\perp$ to NC and such that $NG^2 = NM \cdot Nq$ the locus of the point G will be a right line CG given in position and GB^2 will always be equal to $BN^2 + NG^2 = BN^2 + NM \cdot Nq = BN^2 + Nq^2 + qO^2 = qS^2 + qN^2 + qO^2$, therefore $qS^2 + qN^2 + qO^2$, and consequently $qI^2 + qF^2 + qE^2$ will be a minimum when BG is a minimum, that is when BG is $\perp$ to CG or as above said when $BN \cdot NC = (NG^2) = NM \cdot Nq$.—Hence the following

Demonstration of the Theorem.

Produce AB to Y so that $AB \cdot AY$ may be $=$ to $AB^2 + BC^2 + CA^2$, divide AB in q so that $Aq : qB :: AC^2 : CB^2$, join Cq and demit the $\perp CD$; draw $MQNG \parallel AB$ so that $MN \cdot NQ = BN \cdot NC$, draw Bd, Bq', $Ba \parallel CD$, Cq, CA respectively, join XY meeting BD in g and draw $ngQ'm \parallel ba$. Then from the construction it is evident that $AB \cdot BY = AC^2 + CB^2$ and also (since $Aq : qB :: AC^2 : BC^2$) that $Aq.BY = AC^2$ and $Bq \cdot BY = BC^2$; now $YB : dX$ $(:: Bg : gd :: Bn : nC) :: Bn^2 : Bn.nC :: YB \cdot Bq : AB\ (dX) \cdot Bq :: BC^2 : aC \cdot Cq :: Bn^2 : mn.nq'$ that is $Bn^2 : Bn.nC :: Bn^2 : mn\ nq$; whence $Bn.nC = mn.nq$; therefore $ag = DG$. But $AY \cdot AB : AX \cdot AB :: dX : (dg \cdot AX)\ DG \cdot DC$, that is
$AC^2 + CB^2 + BA^2$: twice the area :: twice the area : the sum of the squares of the $\perp$s demitted from Q, for per the Lemma $qO^2 + qN^2 + qS^2 : qF^2 + qE^2 + qI^2 :: qS^2 : qI^2 :: BN^2 : DG^2 :: BN\ BC : DG \cdot DC\ (dg.AX)$; but $BN \cdot BC = qO^2 + qN^2 + qS^2$, therefore $DG\ DC = qF^2 + qE^2 + qI^2$.

N. B. In this solution reference must be had to both figures all along.

The same, by Mr. John Whitley.

Let ACB (fig. 697, pl. 33.) be any plane $\triangle$, and P a given point within it from which perpendiculars are drawn to the sides. Then since the sum of the squares of those perpendiculars must be

a

a minimum, any two of them may be varied independent of the third; therefore through the point P, draw GH parallel to BC meeting AC, and AB in H and G, and join AP, BP, CP, and GC. It will readily appear, that whenever the point P falls in GH, the fum of the $\triangle$s APC, APB will always be equal to the conftant $\triangle$ ACG, and confequently if one of thefe $\triangle$s, as APB, be increafed by the fmall rectangle $AB \times m$, the other, APC, will have decreafed by the fame rectangle $AB \times m = AC \times n$; that is, if the perpendicular PD, be increafed till it becomes equal to $PD + m$, the perpendicular PE will be decreafed till it becomes equal to $PE - n$; and confequently $PD^2 + PE^2$ will be ultimately equal to
$(PD + m) \times PD + (PE - n) \times PE$
and therefore $PD \times m = PE \times n$;
and becaufe $AB \times m = AC \times n$, we have $PD \times AC = PE \times AB$, or

$$PD = \frac{PE \times AB}{AC}.$$

In the very fame way it may be fhewn that

$$PF = \frac{PE \times BC}{AC}.$$

Now draw CR perpendicular to AB; then

$$CR \times AB = PD \times AB + PF \times BC + PE \times AC$$

$$= \frac{PE \times AB^2}{AC} + \frac{PE \times BC^2}{AC} + PE \times AC;$$

confequently $PE = \dfrac{AC \times CR \times AB}{AB^2 + BC^2 + AC^2}$; wherefore

$$PF = \frac{BC \times CR \times AB}{AB^2 + BC^2 + AC^2};$$

$$\text{and } PD = \frac{AB \times CR \times AB}{AB^2 + BC^2 + AC^2}.$$

Hence $PE^2 + PF^2 + PD^2 = \dfrac{CR^2 \times AB^2}{AB^2 + BC^2 + AC^2}$; or

$AB^2 + BC^2 + AC^2 : CR \cdot AB :: CR \cdot AB : PE^2 + PF^2 + PD^2$,
that is,
$AB^2 + BC^2 + AC^2 : 2\triangle ACB :: 2\triangle ACB : PE^2 + PF^2 + PD^2$.

Q. E. D.

The same, by Mr. James Cunliffe.

Fig. 698, pl. 38. Let Q be the point in the triangle ABC from whence perpendiculars being drawn to the sides, the sum of their squares will be the least possible. Through Q let the right lines DE, FG, and HI be drawn parallel to the sides AB, BC, and AC respectively.

Put $AB = a$, $BC = b$, $AC = c$, and $HF = x$, and let A denote the area of the $\triangle$ ABC. Then $\frac{2A}{AB}$ will express the length of a pependicular from the $\angle$ C upon AB; and, by the property of the point Q, as demonstrated in the Tracts and Selections, in Art. 28 of Playfair's Paper on the Origin and investigation of Porisms.

$AB^2 : AC^2 :: FQ : QG :: HF : DQ = AH$, that is,

$AH = \frac{AC^2 \times HF}{AB^2} = \frac{c^2x}{a^2}$; and for the same reason

$AB^2 : BC^2 :: HF : QE = FB = \frac{BC^2 \times HF}{AB^2} = \frac{b^2x}{a^2}$;

therefore $AH + HF + FB = \frac{c^2x}{a^2} + x + \frac{b^2x}{a^2} = AB = a$;

Whence $x = \frac{a^3}{a^2 + b^2 + c^2}$. Again per similar $\triangle$s

$AB : HF :: \frac{2A}{AB} : \frac{2A \times HF}{AB^2} = \frac{2Ax}{a^2} = \frac{2Aa}{a^2 + b^2 + c^2} =$ the length of a perpendicular from Q upon AB. And in the same way we shall find that

$\frac{2Ab}{a^2 + b^2 + c^2}$ and $\frac{2Ac}{a^2 + b^2 + c^2}$ express the lengths of perpendiculars from Q, upon BC and AC respectively.

And the sum of the squares of the said perpendiculars is

$\frac{4A^2 \times (a^2 + b^2 + c^2)}{(a^2 + b^2 + c^2)^2} = \frac{4A^2}{a^2 + b^2 + c^2}$ from whence it is very obvious that twice the area of the $\triangle$ ABC is a mean pro-

 portional

portional between the fum of the fquares of the perpendiculars from Q, and the fum of the fquares of the fides of the triangle.

The fame, anfwered by Amicus.

LEMMA. *Problem.* In a plain triangle ACB (fig. 66a, pl. 92.) it is required to find a point Q in one fide, fuch that the fum of the nth powers of QF and QE, drawn perpendicular to the other two fides, may be a minimum.

Put $QG = x$, $GH = a$; then $QF = x \times$ fin. G, and $QE = (a - x) \times$ fin. H; and the quantity to be a minimum is $x^n \times$ fin.$^n G + (a - x)^n \times$ fin.$^n H$; putting the fluxion $= 0$ we eafily get $x^{n-1} \times$ fin.$^n G = (a-x)^{n-1} \times$ fin.$^n H$; that is $QG^{n-1} \times$ fin.$^n G = QH^{n-1} \times$ fin.$^n H$. Whence $QG^{n-1} : QH^{n-1} ::$ fin.$^n H :$ fin.$^n G :: GC^n : CH^n$. Therefore the point Q is found by dividing GH in a certain given proportion.

Cor. The $(n-1)$th powers of the perpendiculars QF and QE are as the fides of the triangle on which they fall.

For, fince x fin. $G = QF$ and $(a - x)$ fin. $H = QE$, therefore, becaufe $x^n \times$ fin.$^n G = (a-x)^n \times$ fin.$^n H$, $QF^{n-1} \times$ fin. $G = QE^{n-1} \times$ fin. H. Confequently $QF^{n-1} : QE^{n-1} ::$ fin. H : fin. G :: GC : CH.

Demonftration of the Prize Queftion.

Let ACB be the triangle, and Q the point fuch that the fum of the fquares of QF, QE, and QD is a minimum; and through Q draw GH parallel to one fide as AB. Then, becaufe all the perpendiculars drawn from GH to AB are equal, the fum of the fquares of QF and QE muft be lefs than the fum of the fquares of perpendiculars drawn from any other point in GH to GC and CH. Therefore (Cor. to Lem. putting $n = 2$) QF : QE :: GC : CH :: AC : CB.

But what has been demoftrated of QF, QE, and the fides AC and CB may in like manner be demonftrated of QF, QD and AC, AB, therefore we have thefe two proportions.

QF : QE :: AC : CB
QF : QD :: AC : AB.

Now,

Now, from the proportions just found, we readily derive these two (E. 1. 6)

$$QF^2 : QE^2 :: AC \times QF : BC \times QE$$
$$QF^2 : QD^2 :: AC \times QF : AB \times QD$$

whence (E. 12. 5)

$$\left\{ \begin{array}{l} QF^2 : (QF^2 + QE^2 + QD^2) :: AC \cdot QF : (AC \cdot QF + BC \cdot QE \\ + AB \cdot QD =) \; 2\triangle \; ACB. \end{array} \right\}$$

Again, from the same two proportions, the two following are in like manner derived, viz.

$$QF \times AC : QE \times BC :: AC^2 : CB^2$$
$$QF \times AC : QD \times AB :: AC^2 : AB^2$$

whence (E. 12. 5)

$$\left\{ \begin{array}{l} QF \times AC : (QF \cdot AC + QE \cdot BC + QD \; AB =) \; 2 \triangle \; ACB :: \\ AC^2 : (AC^2 + BC^2 + AB^2) \end{array} \right\}$$

But, if the two proportions that have now been demonstrated be taken alternately, the two first ratios will evidently be equal; therefore also the two latter ratios will be equal; that is

$$QF^2 + QE^2 + QD^2 : 2\triangle ACB :: 2\triangle ACB : AC^2 + AB^2 + BC^2.$$

Q. E. D.

The Medal for solving the Mathematical Prize Question is decided in favour of Mr. I. T. M'DONELD, who will please to send for it to Mr. GLENDINNING'S.

ARTICLE XXI.

Anſwers to the Mathematical Queſtions propoſed in

ARTICLE XV. No. XIII.

I. QUESTION 301, *anſwered by Mr.* Tho. Boole.

The ſide of the octagon being 10 chains its area was 48·28427 acres, and annual rent £96.56854. Therefore, by Dr. Hutton's Theorems for Annuities in arrear, (at page 260, Vol. I. of he Courſe.)

$\frac{1{\cdot}05^{50}-1}{1{\cdot}05-1} \times 96{\cdot}56854 =$ £20216·8*s.* 7½*d.* the ſum required.

Ingenious anſwers were alſo given by Meſſrs. Amos, Collins, Cunliffe, Exton, Francis, Hearding, Morley, I. Smith, Squire, Thompſon, Tyro Philomatheticus, *and* Whitley.

II. QUESTION 320, *anſwered by Maſters* Wm. Amos *and* Wm. Exton, *at Frieſton School, near Boſton.*

Put $a =$ axis, $s =$ ſurface, $S =$ ſolidity, and $p = 3{\cdot}141592$ &c. Then

Firſt, when $s = S$, we have $pa^2 = \frac{1}{6}pa^3$, or $a = 6$;

Secondly, when $a = S$, we have $a = \frac{1}{6}pa^3$, or $s = pa^2 = 6$; and

Thirdly, when $a = s$, we have $a = pa^2$, or $a = \frac{1}{p}$, and

therefore, $S = \frac{1}{6}pa^3 = \frac{1}{6p^2} = {\cdot}016885$.

That is, when the ſuperficies and ſolidity of a globe are equal the axis is 6; when the axis and ſolidity are equal the ſuperficies is 6; and when the axis and ſuperficies are equal, the ſolidity is equal

equal to unity divided by 6 times the ſquare of the circumference of a circle whoſe diameter is unity or 1.

Ingenious anſwers were alſo given by Meſſrs. Boole, Collins, Cunliffe, Francis, Hearding, Morley, Thompſon, Tyro Philomatheticus, *and* Whitley.

III. QUESTION 303, *anſwered by Mr.* Piſhey Thompſon, *of Frieſton School, near Boſton.*

Firſt $70\frac{1}{2}$, the cubic inches in an ale quart, divided by $(4^2 \times \cdot 7854) = 5\cdot 612 = AC$, (fig. 696, pl. 33.) Since ad is perpend. to Cac; by Euc. 13, VI. $Ca \times ac = (ad)^2$, whence $Ca = \cdot 268$. Let ae be drawn parallel to CA;
then will $eb = AB - ae - bB = AB - 2Ae = AB - 2Ca = 3\cdot 464$;
finally $(ab)^2 = (ae)^2 + (eb)^2 = (CA)^2 + (eb)^2 = 43\cdot 494544$;
or $ab = 6\cdot 595$ inches nearly the length of the braſs plate.

The ſame, by Mr. Thos. Morley.

Firſt $\dfrac{70\cdot 5}{4^2 \times \cdot 7854} = 5\cdot 6102$ the depth of the quart. Then $2\sqrt{(2^2 - 1^2)} = 3\cdot 46414$, the diſtance of the lower end of the plate from a point on the bottom directly under the upper end; therefore $\sqrt{(5\cdot 6102^2 + 3\cdot 4641^2)} = 6\cdot 5935$ inches the length of the plate required.

Other ingenious ſolutions were given by Meſſrs. Collins, Cunliffe, Francis, Hearding, I. Smith, Squire, Tyro Philomatheticus, *and* Whitley.

IV. QUESTION 304, *anſwered by Mr.* Tho. Boole.

Fig. 686, pl. 33. Let AB and CD be the poles, EF the chord, G the point in which the poles meet, and GH the perpendicular on the ground, then will ILK be the poſition of the chord when the poles meet;

Now

Now $IL = LK = HD = DK = 5$ yards, and $\sqrt{(5^2 + 3^2)} = \sqrt{34} = DL$;
and DL : radius :: HL : sine $\angle HDL = 30° 57' 49''$,
also KL and KD being equal we have
DK : radius :: $\frac{1}{2}DL$: cosine $\angle LDK = 54° 19' 53''$,
and $HDL + LDK = HDG = 85° 17' 42''$,
and cosine $\angle HDG : HD$ :: radius : $DG = 60.9567$ yards, the length of the poles, as required.

The same answered, by Mr. Tho. Squire, *Baldock.*

Let AB, CD represent the poles, EF the rope, e the middle point thereof, or the first position of the weight.

Let L be the position of the weight when the poles meet at G, bisect BD in H and join HG, DL, and draw LX perpend. to DL meeting DG in X, also draw VX parallel and PX perpend. to DH.

Then because $LK = eF = \frac{1}{2}BD = 5$ and $DF = DK = 5$, we have $LK = DK = KX = 5$ (because $\angle DLX$ is a right one); therefore $DX = 10$, and since $Le = 2$, $HL = 2$; therefore $DL = \sqrt{(HL^2 + DH^2)} = \sqrt{(3^2 + 5^2)} = \sqrt{34}$, and $LX = \sqrt{(DX^2 - DL^2)} = \sqrt{66}$. But by sim. Δ s,

$$DL(=\sqrt{34}):HL(=3)::LX(=\sqrt{66}):VX = 3\sqrt{\frac{66}{34}} = 4.179783;$$

also $DP = DH - VX (= .820217) : DX (= 10) : DH (= 5) : BD$
also $DP = DH - VX : DX :: DH : DG = 60.95944$ yards the length of the poles.

The same, by Mr. William Francis, *Taplow, near Maidenhead.*

Here $\sqrt{(5^2 - 2^2)} = Ie$, and $Ie \times 2 = IK$;
also $\frac{1}{2}(BD - IK) = Bi$; and $Bi : BI :: BH : BG = DG = 60.96$ yards nearly, the answer.

This Question was also answered by Messrs. Collins, Cunliffe, Hearding, Morley, Tyro Philomatheticus, *and* Whitley.

V.

V. QUESTION 305, *anſwered by Mr.* John Collins, *Kenſington.*

Let $p =$ the product of the required numbers, then the ſum of the nth powers will be expreſſed by

$$s^n - n\,(s^{n-2})\,p + n.\frac{n-3}{2}\,(s^{n-4})\,p^2 - n.\frac{n-4}{2}.\frac{n-5}{3}\,(s^{n-6})\,p^3$$

$+ n.\frac{n-5}{2}.\frac{n-6}{3}.\frac{n-7}{4}\,(s^{n-8})\,p^4 -$ &c. taking $n = 9$ we have $512 - 1152p + 864p^2 - 240p^3 + 18p^4 = 32$, or

$$p^4 - \frac{40}{3}p^3 + 48p^2 - 64p + \frac{80}{3} = 0;$$

and this equation reſolved into its factors, becomes

$(p-2) \times (p-2) \times (p^2 - \frac{28}{3}p + \frac{20}{3}) = 0$, and conſeq.

$p^2 - \frac{28}{3}p + \frac{20}{3} = 0$; a quadratic, one root of which is

$\frac{14 - 2\sqrt{34}}{3} = p$. Now from the ſquare of $(x+y)$ take $4p$; and extracting the ſquare root we have $x - y = \sqrt{(s-4p)}$.

Therefore $x = \frac{s + \sqrt{(s^2 - 4p)}}{2}$ and $y = \frac{s - \sqrt{(s^2 - 4p)}}{2}$.

Whence $x = 1{\cdot}4697175$ and $y = {\cdot}5302824$.

Solutions to this queſtion were alſo received from Meſſrs. Morley, *and* Whitley.

VI. QUESTION 306, *anſwered by Mr.* J. H. Hearding.

It is well known that a column of water whoſe height is 33 feet, is equal in weight to a column of air of the ſame baſe, and whoſe altitude is equal to that of the atmoſphere above the earth's ſurface. It has alſo been found by divers experiments, that a body of water is to a body of air of an equal bulk as 850 to 1. If any number

number of altitudes of the atmosphere be taken in an increasing arithmetical progression, the respective densities of the air, at those altitudes, will form a decreasing geometrical progression; or, in other words, the densities of the air at the earth's surface B, and the heights BD, BF (see fig. 685, pl. 33) are to one another as the ordinates AB, CD, EF of the logarithmic curve ACE, whose asymptote is BDF. This is demonstrated by Dr. Halley, in the Philosophical Transactions, No. 181; it has also been very clearly proved by Dr. Gregory, in his valuable Treatise of Astronomy, Prop. 3, Book V. To apply what is here premised to the solution of the proposed question; assume AB = 33 feet, the height of a column of water; BD = 850, the altitude of a column of the atmosphere of equal weight with a column of water of 1 foot high; CD = 32 the density of the air at the height of 850 feet; BF = 30 miles = 158400 feet, the given altitude, and EF = the density of the air at that height; then, since in the present case, AB is supposed to be equal 1, CD will be represented by $\frac{32}{33}$, and,

as 850 (= BD) : 158400 (= BF) or
as 17 : 3168 :: log. (AB ÷ DC) : log. (AB ÷ EF) = 2·4904021, consequently log. EF = − 3·5095979, and EF = 0·0032329, the density of the air at the height of 30 miles when the density at the surface is = 1; let this = d, and put the capacity of the barrel and receiver together = 11 = c, that of the receiver alone = 9 = r, and the number of strokes of the piston, necessary to rarify the air to the given density d, = x; then, by Hutton's Course Vol. II. page 329,

$$x \text{ is} = \frac{\text{Log. } d}{\text{Log. } r - \text{Log. } c} = \frac{-2{\cdot}4904021}{-0{\cdot}0871502} = 28{\cdot}576 \text{ turns of}$$

the piston.

The same, by Mr. John Smith, *Alton Park.*

First, $\frac{30 \times 5280}{63551} = 2{\cdot}4924863$ the logarithm of the rarity of the air at the altitude of 30 miles, when the temperature is 55°.

Then

Then $\frac{2\cdot4924863}{\log.(9+2)-\log.(9)} = 28\cdot6$, nearly the number of strokes required.

See Dr. Hutton's Course of Mathematics, Vol. II. art. 329 and 341.

Ingenious solutions were also given by Messrs. Collins, Cunliffe, Thompson, *and* Tyro Philomatheticus.

VII. QUESTION 307, *answered by Mr.* Francis, *the Proposer.*

Draw the equilateral triangle FBC. (fig. 687, pl. 33.) On FC take FG = 12, and draw GH || to BC. From the centres G and H, with radii 8 and 9 describe arcs intersecting in I, and draw GI. Then if ∠ FCD be made = ∠ FIG, the line CD will meet FI produced in the required point D.

For since the ∠s FCD, FIG are equal and GFI common to both the △s CDF, FIG, the other ∠s must also be equal and the △s similar. If HI be drawn it may be proved in the same manner that the △s FIH, FBD are similar.

Again. draw DK ⊥ and = to BD; bisect KC in L and produce LC to M, making LM = DL. From M, through D draw the line MA, on which demit ⊥s from B and C, then will ABCEDA be the garden required. For if KN be drawn ⊥ to AM the △s ABD, DKN, will be similar and equal.

The △ EMC is also similar to the △MKN;
EM is evidently = DN and MN = DE.
Hence as EM : EC :: MN : KN consequently
$\frac{1}{2}$ DE·EC = $\frac{1}{2}$DN·KN = $\frac{1}{2}$AB·AD.

W. W. R.

CALCULATION. In the △ HIG are given all the sides, whence the ∠s are = 41° 25′, 48° 11′, and 90° 24′. And the ∠ IGF = 108° 11′, the ∠ IHF = 101° 25′, ∠ IFG = 27° 40′, ∠ FIG = 44° 10′, ∠ IFH = 32° 20′, ∠ FIH = 46° 15′, and IF = 16·368. Therefore by sim. △s, as IF (= 16·368) : FC :: IG : DC = 29·32 and :: HI : BD = 32·985. Now we have all the sides of △BCD, consequently the ∠s 14° 43′, 16° 36′, and 148° 41′. Hence ∠ KDC = 58° 41′, ∠ DKC = 54° 41′. and ∠ DCK = 66° 38′.

Also the side KC = 30.697, the ∠DLC = 82° 8′, DL = LM

$LM = 27.171$, whence $CM = 11{\cdot}825$; also $\frac{1}{2} \angle$ DI
$\angle$ CEM $= 41^\circ\ 4'$, whence $\angle$ LDM $=$ LMD $= 48^\circ\ 5$
And therefore $EC = 8{\cdot}9154$, and $ED = 27\ 9316$, ar
area of the garden $= 500$ square yards.

VIII. QUESTION 308, *answered by Mr.* Wm. Marr

Put $\cdot 7854 = p$, and let the conjugate axe be denoted by
$1000 - x$ is the transverse axe, and $1000px - px^2$ is the
the ellipse; also $\frac{1}{2}x + 500$ is its value in guineas: therefo

$$1000px - px^2 : \tfrac{1}{2}x + 500 :: 4840 : \frac{(\frac{1}{2}x + 500) \times 484}{1000px - px^2}$$

the value of one acre, which by the question is to be a mini

or $\dfrac{x + 1000}{1000x - x^2}$, a minimum.

This put into fluxions and reduced, becomes
$x^2 + 2000x = 1000$; whence $x = 414{\cdot}2135624$ the cor
and $1000 - x = 585{\cdot}7864376$ the transverse axis. Hen
area is 39ac. 1r. 20p. nearly.

The same, answered by Tyro Philomatheticus.

Let $x =$ transverse, $y =$ conjugate, $a = 1000$ yards
$p = \cdot 7854$;
Then $pxy =$ the content in square yards,
and $\frac{1}{2}x + y =$ the value in guineas,

therefore $\dfrac{\frac{1}{2}x + y}{pxy} =$ the price per yard, which by the qu

should be a minimum, or $\dfrac{x + 2y}{xy} =$ a minimum.

In fluxions $\dfrac{xy(\dot{x} + 2\dot{y}) - (x + 2y)(x\dot{y} + \dot{x}y)}{x^2y^2} =$

or $x^2\dot{y} + 2y^2\dot{x} = 0$, but $\dot{x} = -\dot{y}$, hence $x^2 = 2y^2$, or

$x = y\sqrt{2}$, and therefore $x = a \times \frac{\sqrt{2}}{1+\sqrt{2}} = 585{\cdot}79$, $y = a \times \frac{1}{1+\sqrt{2}} = 414{\cdot}21$, and the area $= 190569{\cdot}456$ yards.

IX. QUESTION 309, *anſwered by Mr.* John Smith.

The ſurface of a ſphere or hemiſphere is equal to the curve ſurface of its circumſcribing cylinder, therefore $3{\cdot}1416 \times 24 \times 40 = 3015{\cdot}936$ feet $= 335{\cdot}104$ yards, the ſurface of the balloon, or the quantity of ſilk required.

The ſolidity of a ſphere being $\frac{2}{3}$ that of its circumſcribing cylinder $\cdot 7854 \times 23^2 \times 32 - 14476{\cdot}4928$ feet the capacity of the balloon.

Hence $14476{\cdot}4928 \times (1\frac{2}{9} - \frac{1\frac{2}{9}}{13}) = 16332{\cdot}4534$ ounces $= 9{\cdot}11409$ cwt. the weight with which it would aſcend.

Now by Dr. Hutton's Menſuration, Prob. 2, page 269, 3rd. Edit. $(8800)^2 \div 3000 = 25813\frac{1}{3}$ the parameter, and, by Rule 1, Prob. 4. p. 271, ibid.

$$\frac{25813\frac{1}{3}}{2} \times \left\{ \begin{array}{l} \left\{ \sqrt{\left[\left(\frac{17600}{25813\frac{1}{3}}\right)^2 + 1\right]} \right\} \times \frac{17600}{25813\frac{1}{3}} \\ + \text{hyp.log.} \left\{ \frac{17600}{25813\frac{1}{3}} + \sqrt{\left[\left(\frac{17600}{25813\frac{1}{3}}\right)^2 + 1\right]} \right\} \end{array} \right\} =$$

$18954{\cdot}6502$ yards $= 10{\cdot}76968$ miles, the length of the parabolic tract deſcribed by the Aeronaut.

It was alſo ingeniouſly anſwered by Meſſrs. Boole, Cunliffe, Francis, Hearding, Thompſon, *and* Squire.

X. QUESTION 310, *anſwered by* Ztrepmog.

An n ſhould have been put within the firſt radical of the firſt and third expreſſions, and an m in that of the ſecond expreſſion. This being corrected, put x for the value and alſo in that part of the ſurd

surd where the expression recurs instead of continuing the surd which will give an equation for x.

Thus in the first we have

$$x = \sqrt[n]{a + \frac{b}{\sqrt[n]{a + \&c.}}} = \sqrt[n]{a + \frac{b}{x}}, \text{ therefore}$$

$$x^n = a + \frac{b}{x} \text{ and } x^{n+1} - ax = b.$$

In the second we have

$$x = \sqrt[m]{a + \frac{b}{\sqrt[n]{c + \frac{d}{\sqrt[m]{a + \&c.}}}}} = \sqrt[m]{a + \frac{b}{\sqrt[n]{c + \frac{d}{x}}}}$$

$$= \sqrt[m]{a + \frac{bx^{\frac{1}{n}}}{\sqrt[n]{cx + d}}} = \sqrt[m]{\frac{a\sqrt[n]{cx + d} + bx^{\frac{1}{n}}}{\sqrt[n]{cx + d}}}.$$

Therefore $x^m \times \sqrt[n]{cx + d} = a \times \sqrt[n]{cx + d} + bx^{\frac{1}{n}}$.

And in the third we have

$$x = \sqrt[n]{\frac{a}{b + \sqrt[n]{\frac{a}{b + \&c.}}}} = \sqrt[n]{\frac{a}{b + x}}; \text{ therefore } x^{n+1} + bx^n =$$

The chief use of these recurring surds or fractions is for solvi equations which they sometimes do with advantage.

Thus if $x^{n+1} + bx^n = a$, be an equation in which b greater than a we have

$$x = \sqrt[n]{\frac{a}{b + x}} = \sqrt[n]{\frac{a}{b + \sqrt[n]{\frac{a}{b + x}}}} = \sqrt[n]{\frac{a}{b + \sqrt[n]{\frac{a}{b + \sqrt[n]{\frac{a}{b + \&}}}}}}$$

The same as above, which surd will continually approximate the value of x.

XI. QUESTION 311, *answered by* Cap. Geo. Gorry.

Let SCRH (fig. 699, pl. 33.) represent a section of the given sphere passing through the centre O, and ABGI the inscribed cube; then it is manifest that AI the diagonal of the cube is a diameter of the sphere passing through the centre O. Hence to find aI or AG, let it be considered that

$AI^2 = AG^2 + GI^2 = AG^2 + Ga^2 + aI^2 = 3AG^2$; therefore $AI = AG\sqrt{3}$, and $AG = AI \div \sqrt{3} = \frac{1}{3}AI\sqrt{3}$; that is, in the present case, the side of the inscribed cube is $4\sqrt{3} = 6{\cdot}92820323$. Next to find the side cb, bd, or de, of the base of the octagonal pyramid (whose vertex will be V) let it be denoted by x; and since abc is a right angled triangle having $ab = ac$,

it will be $ab : x :: 1 : \sqrt{2}$,

consequently $ab = x \div \sqrt{2} = \frac{1}{2}x\sqrt{2}$

and $ab + bd + dI = aI$;

that is $x + 2 \times \frac{1}{2}x\sqrt{2} = x + x\sqrt{2} = aI$;

hence $x = \dfrac{aI}{1+\sqrt{2}}$, and $x^2 = \dfrac{aI^2}{3+\sqrt{2}} = \dfrac{48}{3+\sqrt{2}} =$ square of a side of the octagonal base.

The solidity of the pyramid will therefore be

$$\frac{48}{3+\sqrt{2}} \times \frac{4\sqrt{3}}{3} \times 4{\cdot}8284271 = 91{\cdot}8322 \text{ \&c.}$$

It remains to find the solidity of the greatest circular spindle, inscribed in the space between Aa, and SCR. Here $CD = OC - OD = 2{\cdot}53589838$, and $CE = 1{\cdot}26794919$; from which, since CH is known, we easily find the area of the generating segment RCSE $= 6{\cdot}38135136$.

Again $ER = \sqrt{(CE \cdot EH)} = \sqrt{\left\{\frac{1}{2}DC \times (1\frac{1}{2}DC + aI)\right\}} = 3{\cdot}6888611$.

Therefore (Dr. Hutton's Mensuration, page 215, Ed. II.)

$$\left\{\tfrac{1}{3}ER^3 - (EO \times \text{area } CER)\right\} \times 4 \times 3{\cdot}141593 = 20{\cdot}5815 \text{ \&c.}$$

solidity of the circular spindle RSCD. And consequently $91{\cdot}8322 - 20{\cdot}5815 = 71{\cdot}3007$ the difference required.

The same, by Mr. John Smith.

By Dr. Hutton's Mensuration, page 186, 3d Edit. $4\sqrt{3} = 6.9282032$ the side of the cube. Then (ibid. pa. 85.) the area of an octagon whose side is 1 is 4·8284271, which divided by 4, half the perimeter, gives 1·2071068 nearly for the perpendicular from the centre to the middle of one of its sides; and $6.9282032 \div 2 = 3.4641016$, the perpendicular from the centre of the base of the octagonal pyramid to the middle of one of its sides; whence by similar figures,

$(1.2071068)^2 : 4.8284271 :: (3.4641016)^2 : 39.764502$ the area of the base; then

$39.764502 \times 6.9282032 \times \frac{1}{3} = 91.832183$ the solidity of the pyramid.

Next, $3.4641016 + \frac{6 - 3.4641016}{2} = 4.7320508$ the central distance of the spindal;

and $\sqrt{(6^2 - 4.7320508^2)} = 3.688861$ nearly, $=$ half its length.

Then by Trigonometry,

$6 : 3.688861 :: 1$ (rad.) $: .6148101 =$ sin. 37.938121 deg. theref.

$$\frac{37.938121 \times .0174533 \times 6}{2} - \frac{3.688861 \times 4.7320508}{2} =$$

3·190678, half the area of the generating segment.

Hence, by the Treatise above referred to, Prob. 21, pa. 158,

$$\left\{\tfrac{1}{3} \times (3.688861)^3 - (3.190678 \times 4.7320508)\right\} \times 4 \times$$

$3.14159265 = 20.531552$ the solidity of the spindle.

Then $91.832183 - 20.531552 = 71.300631$ the difference required.

It was ingeniously answered by Messrs. Collins, Cunliffe, Hearding, *and* Whitley.

XII. QUESTION 312, *answered by Mr.* John Whitley.

Let FCBADF (fig. 674, pl. 33.) represent the garden, and suppose that CED is the longest walk which can possibly be made therein parallel to AB. Then by property of the semicircle

FE : CE :: CE : BE and by a property of the parabola
$DE^2 : AB^2 :: FE : BF$; wherefore
$DE^2 : CE^2 :: AB^2 : BE \times BF = CB^2$, or
BC : AB :: CE : DE; componendo
BC + AB : AB :: DC : DE.

Now AB is constant, and CD is a maximum; therefore (BC + AB) × DE is a maximum, which being fluxed and reduced gives

$$DE = \sqrt{AB^2 - \left\{[\sqrt{(AB^2+8FB^2)} - AB] \times \frac{AB}{4FB}\right\}^2} =$$

27·548; hence CE = 48·406, and therefore CD = 84 yards nearly.

XIII. QUESTION 313, *answered by Mr.* Cunliffe.

ANALYSIS. Fig. 665, pl. 32. Draw the radius BO and from the centre O draw Om perpendicular to AC, and join Bm. Om bisects AC by Euc. III. 3. and
$2(Am)^2 + 2(Bm)^2 = AB^2 + BC^2$, by Simpson's Geometry, Theor. 11, Book II. = a given magnitude by the question; therefore $(Am)^2 + (Bm)^2$ = a given magnitude.

Again $(Am)^2 + (Om)^2 = AO^2 = BO^2$ by Euc. I. 47, and taking the difference between this and the preceding gives
$(Bm)^2 - (Om)^2$ = a given magnitude, whence the locus of m in this case is known to be a right line perpendicular to BO. Now draw PO; since the angle PmO is a right angle the locus of m in this case is known to be the circle whose diameter is PO; wherefore the intersection of these loci will be the point m, through which the line PAC must be drawn.

The same, by Mr. John Whitley.

ANALYSIS. Fig. 676, 677, pl. 33. Let O be the centre of the given circle ABC, join PB, PO, CO and bisect PB in R. Suppose that PAC is really drawn as required.

Let OS be drawn perpendicular to PAC, and join RS.

Then by known properties

$AC = CS$, $\frac{1}{2}AB^2 + \frac{1}{2}BC^2 = BS^2 + CS^2$, $OC^2 - OP^2 = CS^2 - PS^2$; wheref. $\frac{1}{2}AB^2 + \frac{1}{2}BC^2 + OP^2 - OC^2 = BS^2 + PS^2 = 2BR^2 + 2RS^2$, theref. $2RS^2 = \frac{1}{2}AB^2 + \frac{1}{2}BC + OP^2 - OC^2 - 2BR^2 =$ a given ſpace, which take equal to $2Q^2$, and we ſhall then have the following Conſtruction.

Upon PO as a diameter deſcribe the ſemicircle POS, to which from R, apply $RS = Q$; through S draw PAC meeting the circle in A and C, and the thing will be done, as is manifeſt from the Analyſis.

The ſame, by Mr. Swale, the Propoſer.

ANALYSIS. Let PAC (fig. 678, pl. 33.) be ſuppoſed drawn as required. Join AB, BC; biſect AC in G, and join BG, QG, QA, QB, the centre of the circle being at Q.

Now $CB^2 + BA^2$ being given, its equal
$2GB^2 + 2GA^2$, or $2GB^2 + 2QB^2 - 2QG^2$ will be given;
and QA or QB being a given line, it follows that,
$2GB^2 - 2QG^2$ or $GB^2 - QG^2 =$ a given ſpace.

Demit upon BQ the perpendicular GE. Then.
$GB^2 - GQ^2 = BE^2 - QE^2$, or $BE^2 - QE^2 =$ a given ſpace.

Again, $BE^2 - QE^2 = (BE + QE) \times (BE - QE)$
$= (BQ + 2QE) \times BQ$
$= BQ^2 + 2QE \times QB$; conſequently
$2QE \times QB =$ a given ſpace, or QE a given line.

The point E is therefore given, and the lines BE, EG, given in poſition; that is, the point G is in a right line given in poſition; alſo, ſince PQ is given in length and poſition, and QG perpendicular to GP, it follows that the point G is in the circumference of the circle deſcribed upon the diameter PQ; and therefore the point G is given.

Hence the following CONSTRUCTION.

Join the given points P, B with the centre Q, and on each ſide of the point B take, in BQ, BR, BR′ each equal to the ſide of a ſquare equal to the given ſpace; in BQ, make $BQ : QR' = QR : QE$; draw EG perpendicular to EB, meeting a ſemicircle on PQ in G; and draw, through P and G, the required line PAC.

XIV.

XIV. QUESTION 314, *answered by Mr.* Cunliffe.

CONSTRUCTION. Fig. 679, pl. 33. Draw the radius CF perpendicular to AB and join FP. Upon CB take CR such that $2CP \times CR = FP^2$, and upon CR as a diameter describe a semicircle in which apply ST perpendicular to CR such that $4(ST)^2$ may be equal to the given sum of the squares of the perpendiculars.

Again, upon CP as a diameter describe another semicircle, and through O, the point where this latter semicircle cuts the line ST, draw the line required.

DEMONSTRATION. Draw CO which will be perpendicular to, and bisect DE by Euc. 31 and 3. III. also draw the perpendiculars Dm and En.

Because of the similar triangles CPO and COS,
$CP : CO :: CO : CS$, whence $CP \times CS = CO^2$.

Also by reason of the similar triangles PDm and PEn,
$PD^2 : PE^2 :: Dm^2 : En^2$, and by composition
$PD^2 + PE^2 : PE^2 :: Dm^2 + En^2 : En^2$, alternately
$PD^2 + PE^2 : Dm^2 + En^2 :: PE^2 : En^2$
$:: CP^2 : CO^2 :: CP : CS$,
that is $CP : CS :: PD^2 + PE^2 : Dm^2 + En^2$.

Now $PD^2 + PE^2 = DE^2 - 2PD \times PE = 4DO^2 - 2PD \times PE$
$= 4DC^2 - 4CO^2 - 2PD \times PE$;
and $PD \times PE = AP \times PB = DC^2 - CP^2$, wherefore
$PD^2 + PE^2 = 4DC^2 - 4CO^2 - 2 \times (DC^2 - CP^2)$
$= 2DC^2 + 2CP^2 - 4CO^2$
$= 2DC^2 + 2CP^2 - 4CP \times CS$. And by constr.
$2CP \times CR = FP^2 = DC^2 + CP^2$, or
$4CP \times CR = 2DC^2 + 2CP^2$; whence
$PD^2 + PE^2 = 2DC^2 + 2CP^2 - 4CP \times CS$
$= 4CP \times CR - 4CP \times CS$
$= 4CP \times (CR - CS) = 4CP \times RS$. Theref.
$CP : CS :: PD^2 + PE^2 = 4CP \times RS : 4CS \times RS = 4ST^2$;
and from what is deduced above
$CP : CS :: PD^2 + PE^2 : Dm^2 + En^2$, wherefore
$Dm^2 + En^2 = 4ST^2$ the given space by the construction.

The following THEOREM may be gathered from the preceding solution.

In a given circle let the radius CF be drawn perpendicular to the diameter AB, and let the right line FP be drawn to any point P in AB; also let CR be taken a third proportional to twice

CP and FP, and let circles be described upon CP and CR as diameters; then if at any point S between C and P, a right line SOT be drawn perpendicular to AB cutting the peripheries of the circles whose diameters are CP and CR, in O and T respectively; and if through P and O a right line be drawn to cut the given circle in D and E, and Dm, En be drawn perpendicular to AB then will $Dm^2 + En^2 = 4(ST)^2$.

XV. QUESTION 315, *answered by Mr.* Cunliffe.

Fig. 680, pl. 33. Let a circle be described about the triangle, and bisect the base AB by the perpendicular MN meeting the circumference of the circumscribing circle below the base in N, and draw NC. With the centre N and radius NB describe another circle cutting NC in E, which will be the centre of the circle inscribed in the triangle ABC as is very well known; wherefore from E demit E*a* perpendicular to AB and it will be the radius of the inscribed circle, and *a* the point of contact thereof with the base AB. Produce CN till it cuts the circumference of the circle whose centre is N in F, and draw F*b* perpendicular to AB.

Then because NE = NF, M*b* must be equal to M*a*; and therefore B*b* = A*a*. Again by Euc. 35, III.
LN × LC = AL × LB = LE × LF, whence
LN : LF :: LE : LC; and because of the ∥s NM, E*b*,
LN : LF :: LM : L*b*, therefore
LE : LC :: LM : L*b*, wherefore C*b* and EM being drawn will be parallel to each other, from whence and what is before laid down the truth of the proposition will be manifest.

The same, by Mr. J. Collins, *Kensington.*

Produce ME to meet the perpendicular, let fall from the vertical angle upon the base, in P (fig. 681, pl. 33.)

It is evident that the triangles BMF and CDE are similar, and so are the triangles FME and CPE (DE being perpendicular to BC). Hence
BF (EF) FM : CE : CP :: CE : DE;
therefore CP = DE = E*g*, and E*g*CP is a parallelogram;
wherefor

wherefore ME and bgC are parallel, and
$bM : Ma :: Eg : Ba$, and therefore $bM = Ma$;
Wherefore $Bb + bM = Ma + aA$, that is, $BM = MA$.

Q. E. D.

The same, by Mr. John Dawes, *Birmingham.*

Fig. 697, pl. 31. Let fall the perpendiculars Ea, CK, and bM, continue ME till it meets CK in p. Now it is well known that $Cp = Ea$; and because $CpMl$ is a parallelogram $Cp = lM = Eb$; but $Ba = Ab$, and the triangles blM, MEa are similar and equal, therefore M bisects ba, consequently it bisects AB, the base of the triangle.

Q. E. D.

The same, by Mr. Thomas Morley.

Fig. 681, pl. 33. Cb being joined will meet the circumference of the inscribed circle in the same point as aE produced to the circumference, as at g, by Prop. 58, Student. Therefore, because $aE = Eg$, and $EM \parallel bg$, $bM = Ma$; consequently BM is equal MA.

Q. E. D.

The same, by Mr. Whitley.

Fig. 673, pl. 32. Bisect AB in M, and join Ca, EM; produce ME to meet Ca in P. Then because $AM = BM$, and Aa (or $AM + aM$) $= Bb$ (or $BM + bM$), we have $aM = bM$. But it is now pretty well known that $CP = aP$ (see Gent. Diary, quest. 5, 1801) Wherefore the two sides Ca, ba of the triangle Cab are bisected in the points P and M respectively; therefore PEM is parallel to Cb; that is, EM drawn parallel to Cb bisects the base AB in M.

This proposition was also demonstrated by Limentis, *the Proposer.*

XVI.

XVI. QUESTION 316, *answered by* Astronomicus.

It is manifest that the duration of twilight is different at the same time of the year, in different climates of the earth; these differences are clearly stated in a few words in Art. 107, of O. Gregory's astronomy. Thus, "If a place be situated in a parallel, or nearly so, the apparent motion of the sun being either quite or nearly parallel, he will be carried round for some months at less depression than 18°, during which there will, of course be no real night. In a *right sphere*, the twilight is shortest, because the sun rising and setting at right angles to the horizon he is depressed 18° below the horizon in the shortest period. In any place in an *oblique sphere*, the nearer it is to one of the poles, the longer the twilight; and consequently, the nearer the equator, the shorter the twilight."

The twilight then being absolutely a minimum at the equator at all times of the year, it is easy to find its duration; for at the time of the equinoxes it is nothing more than to find in what time 18° are passed over by the sun in a right sphere, the whole circle or 360° being passed over in 24 hours; thus, as 360° : 18° :: 24h. : $\frac{24 \times 18}{360} = \frac{6}{5} =$ 1h. 12m. duration of shortest twilight at the equinoxes.

And at any other time of the year let P (fig. 666, pl. 32, represent the pole, PH the horizon, being a secondary to the equator in a right sphere, H the sun's place, and HC = 18°, the distance from the horizon to the crepusculum circle. Then PH = 90° — sun's decl. and as H is a right angle we have, sine PH : radius :: tang. HC : tang. P. This analogy when the sun's decl is 7° 9', as it is, on Oct. 12th. gives 18° 8' nearly for the angle P; and this converted to time gives 1h. 12m $\frac{8}{15}$. for the duration of the twilight at the equator on October 12th.

As to the duration of the twilight on the same day in lat. 52° 12' N. it is given at Pa. 94, O. *Gregory's Astronomy*, and at Pa. 19, *Professor Vince's Astronomy*; both of whom state it at about 1h 58m $\frac{2}{3}$. So that the duration of twilight at the equator (which is *absolutely* the shortest twilight) is shorter than that in lat. 52° 12' N. by 46m $\frac{2}{3}$.

Remark. Hence appears the mistake of those who conceive that the common analogy for the shortest twilight is applicable to questions like this:—for, although that analogy enables us to find the

the sun's declination when the twilight is a *minimum* at any given latitude, yet the converse of the analogy will not assist us in finding the latitude *where* the twilight is a *minimum*, and the sun's declination given:—the problems are not one and convertible, but two and distinct.

XVII. QUESTION 317, *answered by Mr.* Cunliffe.

It is pretty obvious from what Mr. Simpson has delivered on the maxima and minima of geometrical quantities in the Scholium to Theorem 8, that the sides of the given triangle will be bisected at the points of contact of the greatest inscribed ellipsis. This being noticed, let ABC (fig. 654, pl. 82.) be a given triangle with its greatest inscribed ellipsis touching the sides AB, AC and BC in the middle points D, E, and F respectively; join EF and draw CD intersecting it in d; also draw AF intersecting CD in O. Because the sides AC, BC, are bisected in E and F, the line EF is parallel to AB, which is a tangent to the curve at D, and therefore EF is bisected in d; wherefore CD passes through the centre of the ellipsis, as appears from the properties of the Conic Sections. And for the same reason AF will pass through the centre of the ellipsis, and consequently O, the intersection of CD and AF will be that centre, and OD a semi-diameter. Now between O and C take OH = OD, then will DH be a diameter of the ellipsis.

And because of the parallels AB, EF,
$CF : CB :: 1 : 2 :: dF = dE : AD :: Od : DO$, that is,
$Od = dH : DO :: 1 : 2$, and by compos. $Dd : DO :: 3 : 2$;
but $dH : DO :: 1 : 2$, as just deduced,
whence $Dd \times dH : DO^2 :: 3 : 4$.

Let the semi-diameter OG be drawn parallel to AB; then by the property of the ellipsis

$$Dd \times dH : DO^2 :: dE^2 = \frac{AB^2}{4^2} : OG^2; \text{ wherefore}$$

$$3 : 4 :: \frac{AB^2}{4^2} : OG^2 = \frac{AB^2}{3 \times 4},$$ whence the semi-conjugate diameters are known and the ellipsis may be described by prob. 77. B. 1. Emerson's Conics.

The demonstration of the following THEOREM may be deduced from the foregoing solution.

The sum of the squares of two conjugate diameters greatest ellipsis that can be inscribed in any plane triangle to two-ninths of the sum of the squares of the three side triangle.

For, $OD = \frac{1}{3}$ CD as is well known, and

$CD^2 = \frac{AC^2 + BC^2}{2} - \frac{AB^2}{4}$, by Simp. Geo. 11. II. w

$OD^2 = \frac{CD^2}{9} = \frac{AC^2 + BC^2}{18} - \frac{AB^2}{36}$, and from what been deduced

$OG^2 = \frac{AB^2}{12}$; by adding these together

$OD^2 + OG^2 = \frac{AC^2 + BC^2 + AB^2}{18}$, and therefore

$4OD^2 + 4OG^2 = \frac{2}{9} \times (AC^2 + BC^2 + AB^2)$.

Q. E. D

The same, by Mr. John Whitley.

Let ACB (fig. 656, pl. 32.) be the given triangle, bisect AB in D, and join CD; suppose D*n*S*m* the greatest inscribed ellipsis; draw the conjugate diameter *nom* and the tangent ST parallel to AB, draw also SE parallel to BC.

Now by similar triangles

DC : DB :: DS : DE, alternately

$DC : DS :: DB : DE :: DB^2 : EDB = (DB - ST) \cdot DB = DB^2 - DB \cdot ST$

But by Emerson's Conics, Prop. 47, on the ellipsis,

$DB \times ST = om^2$; wherefore

$DC : DS :: DB^2 : DB^2 - om^2 :: DC \cdot om : DS \cdot om =$ a maximum by the question. Consequently

$DB^2 \cdot DS \cdot om = (DB^2 - om^2) \cdot DC \cdot om$ will be a max. but DC is constant, therefore when DS *om* is a maximum, the solid $(DB^2 - om^2) \cdot om$, contained under one leg of a right angled triangle and the square of the other will be a maximum; which is the case when $om^2 = \frac{1}{3} DB^2$ (See Simpson's Geom. Theo. 18. max et min.). Hence $DS = \frac{2}{3} DC$.

Now

Now let $s =$ sine of the angle CDB; then will

$$DS \times nm \times s = \tfrac{2}{3}DC \times 2\sqrt{\tfrac{1}{3}DB^2} \times s = \frac{4DC \cdot DB's}{9}\sqrt{3} = \text{area},$$

and $nm^2 + DS^2 = \frac{4DB^2}{3} + \frac{4DC^2}{9} =$ sum of the squares of the sides of a parallelogram circumscribing the ellipsis $Dn\,Sm$; let the former be taken equal to 2A and latter equal to 4B; also let T = the transverse and C = the conjugate diameters of the ellipsis. Then per Conics, $T \times C = 2A$, and $T^2 + C^2 = 4B$; wherefore

$T = \sqrt{(B + A)} + \sqrt{(B - A)}$ and $C = \sqrt{(B + A)} - \sqrt{(B - A)}$, which are general Theorems for ascertaining the principal diameters of the greatest ellipsis, that can be inscribed in a given triangle.

Cor. 1. If the ellipsis touch the sides of the triangle in D, Q, and R, then will $AR = CR$, and $BQ = QC$.

Cor. 2. The point O which is the centre of the ellipsis is the centre of gravity of the triangle.

XVIII. QUESTION 818, *answered by Mr.* Cunliffe.

Let $a^2 - 1$, and $b^2 - 1$ denote the two numbers, for each of these when increased by 1 will manifestly be a square; but their sum and difference also when increased by 1 must be squares; that is $a^2 + b^2 - 1$, and $a^2 - b^2 + 1$ must each be a square.

Put $a^2 + b^2 - 1 = c^2$, and $a^2 - b^2 + 1 = d^2$, and by adding these two together $2a^2 = c^2 + d^2$, from whence it appears that c^2, a^2, and d^2 are three squares in arithmetical proportion. Therefore we may put

$c = r \times (m^2 - n^2 + 2mn)$, $a = r \times (m^2 + n^2)$, and

$d = r \times (n^2 - m^2 + 2mn)$; but $a^2 + b^2 - 1 = c^2$; whence

$b^2 = c^2 - a^2 + 1 = 4r^2mn \times (m^2 - n^2) + 1$, by writing for c and a their foregoing values.

Assume $b = 2rs - 1$, then

$b^2 = (2rs - 1)^2 = 4r^2s^2 - 4rs + 1 = 4r^2mn \times (m^2 - n^2) + 1$.

Whence

Whence $r = \frac{s}{s^2 - mn \times (m^2 - n^2)}$, where s, m, ar

may be taken at pleasure.

EXAMPLE I.

Take $m = 2$ and $n = 1$, then $r = \frac{s}{s^2 - 6}$, whenc

vious enough, that if we take $s = 3$, then $r = 1$; what has been laid down

$a = r \times (m^2 + n^2) = 5$, and $b = 2rs - 1 = 5$, and $a^2 - 1 = 24$, and $b^2 - 1 = 24$, in which case the tw are equal to each other.

EXAMPLE II.

Take $m = 3$ and $n = 2$, then $r = \frac{s}{s^2 - mn \times (m}$

$\frac{s}{s^2 - 30}$ where it is pretty plain that if we take $s = 6$, th

and hence

$a = r \times (m^2 + n^2) = 13$, and $b = 2rs - 1 = 11$; and hence again $a^2 - 1 = 168$ and $b^2 - 1 = 120$, b numbers that will answer.

EXAMPLE III.

Take $m = 4$ and $n = 3$, then $r = \frac{s}{s^2 - 84}$, where

taken $= 9$, then $r = -3$; and

$a = r \times (m^2 + n^2) = -75$ and $b = 2rs - 1 = -5$ and hence $a^2 - 1 = 5624$ and $b^2 - 1 = 3024$, which Bonnycastle's numbers.

EXAMPLE IV.

Take $m = 5$ and $n = 3$, then $r = \frac{s}{s^2 - 240}$, whe

not difficult to perceive, that if we take $s = 16$, then and

$a = r \times (m^2 + n^2) = 34$ and $b = 2rs - 1 = 31$;
and from these $a^2 - 1 = 1155$ and $b^2 - 1 = 960$, which are another pair of numbers that will anſwer.

Very general formulæ for whole numbers that will anſwer the queſtion may be obtained as follows. We have before found that

$$b^2 = 4r^2 mn \times (m^2 - n^2) + 1 = 4r^2 m^3 n - 4r^2 mn^3 + 1$$

aſſume $b = 2r^2 mn^3 - 1$, then

$$b^2 = (2r^2 mn^3 - 1)^2 = 4r^4 m^2 n^6 - 4r^2 mn^3 + 1$$
$$= 4r^2 m^3 n - 4r^2 mn^3 + 1.$$

Whence $m = r^2 n^5$, $b = 2r^2 mn^3 - 1 = 2r^4 n^8 - 1$, and
$a = r \times (m^2 + n^2) = r \times (r^4 m^{10} + n^2) = rn^2 \times (r^4 n^8 + 1)$,
where r and n may be taken at pleaſure.

EXAMPLE I.

Suppoſe $r = 2$ and $n = 1$, then $a = 34$ and $b = 31$, which agrees with what was obtained in the fourth example, by the firſt method of ſolution.

EXAMPLE II.

Suppoſe $n = 2$ and $r = 1$, then $a = 1028$ and $b = 511$, whence $a^2 - 1 = 1056783$ and $b^2 - 1 = 261120$, which are two other numbers that will anſwer.

Ingenious ſolutions were alſo received from Meſſrs. Boole, Collins, Hearding, *and* Whitley.

XIX. QUESTION 319, *anſwered by Mr.* Gregory.

Fig. 691. Pl. 33. Let Vv be a ray from the ſun, touching V the vertex of the cone VABD, and v the extremity of the ſhadow. Let RGSH, RGSH, &c. be ſections of the cone, whoſe axes RS, RS, &c. are parallel to Vv, and whoſe whole planes coincide with rays from the ſun; theſe ſections will manifeſtly be ellipſes ſo long as the ſun's alt. is leſs than the angle ABV of the

cone, and in no other case can there be a shadow. Prod RS, RS, &c. to o, o, &c. and draw gh, gh, &c. perpendic to Av, and respectively equal to GH, GH, &c. the conjugate of the several ellipses: then (since the solar rays may, so fa relates to this case, be reckoned parallel) will each of these elli be projected into parallel right lines, whose extremities ar gh, gh, &c. Now draw VOOO bisecting all the transverse a RS, RS, &c. which, as these axes are parallel to each other, obviously be a right line. And since HG, HG, &c. are si chords in circles whose dimensions increase as their respective tances VO, VO, &c. from the vertex they will be proportio to those distances: that is,

VO : VO :: HG : HG :: hg : hg. But by reason of the parallels Vv, Ro, &c. it is

VO : VO :: vo : vo; consequently vo : vo :: hg : hg.

And since this is the case in every parallel position of RGE and the corresponding projections hg, hg, &c. it is evident they are the bases to a series of similar triangles, whose comm vertex is v, and whose other sides are $vggg$, $vhhh$; that is, i evident that $vggg$, $vhhh$, the lines bounding the shadow are ri lines. But lines bounding the shadow, cannot, as is clear from nature of such projections, pass *at a distance* from the base of cone, as vI, vK; and they cannot, from what is done above, other than right lines: therefore, *the lines bounding the shadow a cone, standing in the sun-shine, are right lines drawn from the ve tex of the shadow, and touching the base of the cone.* *Q. E. D.*

XX. QUESTION 320, *answered by Mr.* Cunliffe.

The construction of the question will be more neat if the f lowing problem be first laid down as a Lemma.

In a given circle, whose centre is O, (fig. 688. pl. 33.) to dr a chord AB parallel to a right line OQ given by position, so t drawing OM perpendicular thereto, the sum of AB and MO n be of a given length or a maximum.

CONSTRUCTION. Take OP perpendicular to OQ, equal to the given sum of the chord, and perpendicular then from the centre O; also take OQ = $\frac{1}{2}$OP, and draw PQ cutt the circle in B, B′ then through either of the points B, B′ draw chord BA parallel to OQ, cutting OP in M, and the thing done.

DEMO

DEMONSTRATION. Because $OQ = \frac{1}{2}OP$, and OQ and MB are parallel, therefore $MB = \frac{1}{2}PM$, or $2MB = AB = PM$, whence $AB + MO = PM + MO = PO$ the given length by the construction.

Limit. When PO the sum of the chord and perpendicular is a maximum, PQ will just touch the circle in B, and the radius OB will be perpendicular thereto by Euc. 18. III. hence and from what has been deduced $PB = 2OB$ from whence the line PB in the limiting case may be easily drawn.

The construction of the original question may be as follows:

The sum of the squares of the sides, by the question, has a given ratio to the rectangle under the perpendicular and a given line; Wherefore it is pretty obvious, that the sum of the squares of the sides must be equal to the rectangle under he perpendicular and a *given line*, which given line let be denoted by 4L.

CONSTRUCTION. Take a right line O*d* (fig. 689. pl. 33.) equal to half the given difference of the segments of the base, and make the right angled triangle *d*OG such that the hypothenuse $dG = L$; then with the centre O, and radius OG describe a circle, in which by the preceding problem draw the chord BA parallel to O*d* so that $AB + MO$ may be a given length or a maximum.

Through *d* draw *d*D parallel to OM meeting AB in D, and in D*d* produced take $dC = L$, join CA, CB and ACB is the required triangle.

DEMONST. Draw M*d*, MC and the radius OB.

$$\text{Then } \overline{MB}|^2 = \overline{OB}|^2 - \overline{OM}|^2 = \overline{OG}|^2 - \overline{OM}|^2$$

$$= \overline{dG}|^2 - \overline{dM}|^2 = \overline{dC}|^2 - \overline{dM}|^2$$

by Euc. 47. I. and Simpson's Geometry, theor. 9. b. 2.

$$\text{Also } \overline{MC}|^2 = \overline{dC}|^2 + \overline{dM}|^2 + 2dC \times dD \text{ by Euc. 12. II.}$$

and adding the two foregoing expressions together,

$$\overline{MB}|^2 + \overline{MC}|^2 = 2\,(dC)^2 + 2dC \times dD$$

$= 2dC \times DC = 2L \times DC$: from whence and theorem 11. b. 2. Simpson's Geometry.

$$2(MB)^2 + 2(MC)^2 = \overline{AC}|^2 + \overline{BC}|^2 = 4L \times DC.$$

Moreover $BA + Dd$ is a given length, or a maximum by the construction.

Therefore $BA + Dd + dC = BA + DC$ is a given length, or a maximum, because *d*C is a given length.

XXI. QUESTION 321, *anſwered by Mr.* Whitley.

Let ACB (fig 690. pl. 33.) be any plane triangle, AE, BF and CD lines drawn from the angles to biſect the oppoſite ſides and interſecting each other in Q, and let QI, QG and QH be drawn perpendicular to the ſides AB, AC and BC. Then will the ſolid QI·QG QH be a maximum, and three times the ſum of the ſquares of AQ, BQ, CQ equal to the ſum of the ſquares of AC, BC, AB.

For draw FE, DF and DE, then becauſe CF = AF and CE = BE, CE : CF :: BE : AF :: CB : AC, therefore FE is parallel to AC. In the ſame way it is ſhewn that DF is parallel to BC and DE to AC.

Wherefore FE = DB = AD, DF = BE = CE, and DE = AF = CF.

Again, by ſimilar triangles FE : QE :: AB : AQ,
hence it follows that as AB is = 2EF,
AQ will be = 2QE, and therefore AQ = $\frac{2}{3}$AE.
In like manner it is ſhewn that
BQ = $\frac{2}{3}$BF, and CQ = $\frac{2}{3}$DC.

Wherefore $AQ^2 + BQ^2 + CQ^2 = (AE^2 + BF^2 + DC^2) \times$

$$= \left\{ \begin{array}{l} \text{by Simpſon's Geometry II. 11, } (\frac{1}{2}AC^2 + \frac{1}{2}BC^2 - BE^2) \\ + (\frac{1}{2}AB^2 + \frac{1}{2}BC^2 - AF^2) + (\frac{1}{2}AC^2 + \frac{1}{2}BC^2 - AD^2) \end{array} \right\} \times$$

$$= (AC^2 + BC^2 + AB^2 - BE^2 - AF^2 - AD^2) \times \frac{4}{9}$$

$$= (\tfrac{3}{4}AB^2 + \tfrac{3}{4}BC^2 + \tfrac{3}{4}AC^2) \times \frac{4}{9} = \tfrac{1}{3}AB^2 + \tfrac{1}{3}BC^2 + \tfrac{1}{3}AC$$

Conſequently $3AQ^2 + 3BQ^2 + 3CQ^2 = AC^2 + BC^2 + AB$

Again, let CS be drawn perpendicular to AB;
then by ſimilar triangles CS : QI :: CD : QD;
but DQ = $\frac{1}{3}$CD (by above), theref. alſo QI = $\frac{1}{3}$CS.
Wherefore the △ AQB is equal to $\frac{1}{3}$ of the △ ACB.

In like manner it may be ſhewn that the triangles AQC, BQ are each equal to one-third of the triangle ACB, and conſequent the triangles AQB, AQC, and BQC, are equal to each othe therefore 2AQB × 2AQC × 2BQC is, by Simpſon's Geometr Theorem 16, a maximum;

That is (DB·QI) × (FC·GQ) × (CE·QH) = a maximum, a becauſe DB, FC, CE, are here ſuppoſed given lines, the ſolid co tained under QI × QG × QH is alſo a maximum. *Q. E. I*

XXI

XXII. QUESTION 322, *answered by Mr.* W. Wallace.

It is evident that the centre of the given circle is such a point as satisfies the conditions of the question; but let us investigate whether another point can be found, such as is required.

ANALYSIS. Suppose Q (fig. 671, pl. 32.) to be the point which may be found and that DQ produced cuts the circle again in A, then by hypothesis the rectangle CQ × QD is given; but Q being a determinate point, from the nature of the circle AQ × QD is given, therefore the ratio of CQ × QD to AQ × QD is given and hence CQ must have to AQ a given ratio.

Join PQ cutting the circle in B and E; if we now suppose the point C to approach to B, the point A will also approach to B, so that when CQ coincides with BQ, AQ also coincides with BQ; thus it appears that the given ratio of CQ to AQ can be no other than that of equality, and since the line BQ must lie between AQ and CQ, however near these lines may be to each other, it is evident that BQ must pass through F the centre of the circle; thus the arch AB is equal to the arch BC, and if AF, CF be drawn to the centre, the angle AFB is equal to CFB therefore the angle AFC is double the angle AFB or AFP; but AFC being an angle at the centre is also double the angle ADC or ADP at the circumference, therefore the angles AFP, ADP are equal, hence the points A, F, D, B are in the circumference of a circle and the rectangle PQ × QF is equal to AQ × QD, that is to BQ × QF, to each of these rectangles add QF^2, and we have PF × FQ = BF^2. The point Q may therefore be found by the following

CONSTRUCTION.

Let F be the centre of the given circle, and P the given point. Join PF and take the point Q between P and F so that FQ may be a third proportional to PF and FB the radius of the circle.

The same, by Mr. Cunliffe.

Through the centre O (fig. 692, pl. 33.) draw the right line PAB cutting the circle in A and B; then upon PO take PQ a fourth proportional to PO, PB, and PA, and Q is the point.

Demonstration. Draw the radii CO, DO; then by construction $PO : PB :: PA : PQ$, whence $PO \times PQ = PA \times PB$; but by Euc. 36. III. $PA \times PB = PC \times PD$, and consequently $PO \times PQ = PC \times PD$, and therefore a circle will pass through the points C, Q, O, D. Therefore $\angle PDQ = \angle POC$ by Euc. 21. III, and consequently $\angle PQD = \angle PCO$. Also $\angle CQO + \angle PDO = \angle CQO + \angle OCD = \angle PCO + \angle OCD = 2$ right angles by Euc. 13. I, and 22. III, and taking away $\angle OCD$ leaves $\angle CQO = \angle PCO = \angle PQD$, and consequently $\angle QCO = \angle QPD$; therefore the triangles CQO and PQD are similar, therefore $PQ : DQ :: CQ : QO$, whence $CQ \times DQ = PQ \times QO$ a given space.

Mr. Whitley *answered it.*

XXIII. QUESTION 323, *answered by Mr.* Cunliffe.

Fig. 693, pl. 33. By a well known property of triangles $(AC - BC)^2 = 4DQ \times EF$, and therefore the solid $(AC - BC)^2 \times DC = 4DQ \times DC \times EF$, and when this solid is a maximum, the rectangle $DC \times EF$ will be a maximum, because $4DQ$ is given; also when the rectangle $DC \times EF$ is a maximum, its square $DC^2 \times EF^2$ must obviously be a maximum.

Draw FC; then by Euc. 13. II. $DC^2 + 2DF \times EF = DF^2 \times FC^2$, and by a well known property

$FQ \times EF = 2OF \times EF = FC^2$,

the difference of which is

$DC^2 + 2OD \times EF = DF^2 = 2OD \times L$, by taking the line L a third proportional to $2OD$ and DF; whence

$DC^2 = 2OD \times L - 2OD \times EF = 2OD \times (L - EF)$.

Therefore $DC^2 \times EF^2 = 2OD \times (L - EF) \times EF^2$,

and when this expression is a maximum,

$(L - EF) \times EF^2$ will be a max. because $2OD$ is given.

Now when $(L - EF) \times EF^2$ is a maximum, $EF = \frac{2}{3}L$, as appears from Theorem 17th Simpson on the maxima and minima; therefore $3EF = 2L$.

Whence $3DO \times EF = 2DO \times L = DF^2$.

Q. E. D.

The

The same, by Mr. Whitley, *of Attercliffe.*

Let fig. 694, pl. 33, be constructed as per quest. and join AF, CF, AO, AQ, and to the point A draw the tangent AP meeting the diameter FQ in P, also bisect PD in S. Now because the angles PAO, QAF are right angles (Euc. 31, 18, III.) the rectangles $PD \times DO$, $QD \times DF$ are each equal to the square of AD (Euc. Cor. 8, VI.)
therefore the rectangle $PD \times DO = QD \times DF = 2DS \times DO$,
add twice the rectangle $DF \times DO$, to each,
then $(QD+2DO) \times DF = (2DS+2DF) \times DO = 2FS \times DO$;
but $QD + 2DO = DF$, therefore $DF^2 = 2FS \times DO$.

Now let GC be taken equal to the difference of the sides AC, BC; then it will be manifest from what is shewn in Prop. 98. of Simson's Data that
$AF : AB :: CF : CG$, and therefore
$AF^2 : AB^2 :: CF^2 : CG^2 :: CF^2 \times CD : CG^2 \times CD$.

Now since AB and QF are constant, by the hypothesis, AF is so too, and therefore when $CG^2 \times CD$ is a maximum, $CF^2 \times CD$ will be so likewise; but (by Cor. Euc. 8, VI) the rectangle $QF \times EF = CF^2$; wherefore $CF^2 \times CD = QF \times EF \times CD$, and, because QF is constant, when the solid $CF^2 \times CD$ is a maximum, the rectangle $EF \times CD$, and consequently $EF^2 \times CD^2 = 2OD \times ES \times EF^2$ (by Prop. VI. Dr. Stewart's General Theorems) will be so too; and therefore since 2OD is constant, the solid $EF^2 \times ES$ will be a maximum, which is the case when when $EF = 2ES$ (vide Simpson's Geometry, Theo. 17, Max. et Min.);
wherefore $3EF = CES = 2FS$, and $3EF \times OD = 2FS \times OD$;
but it has been shewn that $DF^2 = 2FS \times OD$;
wherefore $3EF \times OD = DF^2$.

Q. E. D.

XXIV. QUESTION 324, *answered by Mr.* John Whitley.

Let N*nm*M (fig. 695, pl. 33.) be a conic section, and let the lines be drawn as per question, also let PM, PN be drawn, the latter meeting AB in *r*, and join *nm*. Now by conics *nm* is parallel to AB. And by similar triangles

vn

$vn : vC :: nN : NM$, alfo $vC : vN :: nm : pN$, wheref. $vn : vN :: nm : NM :: Pn : PN$ (by fim. triangles).

That is the ftraight line PN is harmonically divided by the parallels nm, AB, in the points n, v; and therefore by Cor. 2. to Props. 71, 64, 38, Books 1, 2, and 3, of Emerfon's Conics, the ftraight lines joining the points PA, PB, will be tangents to the Conic Sections at A and B.

The fame, anfwered by Limenus, *the Propofer.*

LEMMA 1. Let lines be drawn from the extremities of any ordinate to a conic fection through a point in the diameter; and the line joining their interfections with the curve will be parallel to the ordinate.

LEMMA 2. Let MN (fig. 652, pl. 32) and mn be two parallel lines, and let Mm and Nn meet in P, and Mn and Nm in C, then PC being joined will bifect the parallels Mm and Nn.

Join PC to meet MN in D, and it follows from Lemma 1, that mn drawn will be parallel to MN, and therefore PC bifects MN by Lemma 2, and P is confequently in the diameter appertaining to AB and MN. Alfo becaufe AC is parallel to MN and CD bifects MN in D, the lines AC, MC, DC, and NC are harmonicals; and they interfect PM in P, m, d, M, and PN in P, n, b, N; therefore Pa, and Pb are harmonical means between the fegments PM. Pm, and PN, Pn. And therfore all other right lines drawn from the point P to AB will be harmonicals means between the fegments intercepted by P and the curve (Maclaurin De Lin. Geom.). But PA meeting AB and the curve in the fame point A, the other point of interfection of PA with the curve neceffarily coincides with the firft; and therefore PA is a tangent at A; and for the fame reafon PB touches the curve in B.

Mr. Cunliffe *alfo favoured us with an ingenious demonftration of this Propofition.*

XXV. QUESTION 325, *anfwered by Mr.* Cunliffe.

In fig. 667, pl. 32, take any two points B and a in the diameter EF of a femicircle and draw from thence the right lines BC, aC, to any point C in the circumference; draw the radius OC and demit CD perpendicular to EF; alfo upon EF take AD =

aD

aD, and join AC. Then Ba is evidently the difference of the segments of the base made by the perpendicular CD in the triangle ACB.

Take the line L such that the rectangle $OB \times L = OC^2 - OB \times Oa$, then
$OC^2 = OB \times (L + Oa)$; whence by Euc. 12. II.

$$BC^2 = OC^2 + OB^2 + 2OB \times OD$$
$$= OB \times (L + Oa) + OB^2 + 2OB \times OD, \text{ and}$$
$$aC^2 = AC^2 = OC^2 + Oa^2 + 2Oa \times OD$$
$$= OB \times (L + Oa) + Oa^2 + 2Oa \times OD$$
$$= OB \times L + OB \times Oa + Oa^2 + 2Oa \times OD.$$

Now let S denote a space to which the square of BC has the ratio of DB to Oa, that is

$$OB : Oa :: BC^2 = OB \times (L + Oa) + OB^2 + 2OB \times OD$$
$$: Oa \times (L + Oa) + OB \times Oa + 2Oa \times OD$$
$$= Oa \times L + Oa^2 + OB \times Oa + 2Oa \times OD$$
$$= S, \text{ whence}$$
$$AC^2 - S = OB \times L - Oa \times L = (OB - Oa) \times L$$
$$= Ba \times L,$$ and hence the following construction is derived.

Take the right line Ba equal to the given difference of the segments of the base and bisect the same in m, in Ba take OB to Oa in the given ratio mentioned in the question, and O will be a given point. Again take a right line L such that the rectangle $Ba \times L$ may be equal to the given excess of the square of one side above the space to which the square of the other side has a given ratio; then in OB take OF such, that $OF^2 = OB \times (L + Oa)$; with the centre O and radius OF describe a semicircle ECF the circumference of which will be the locus of the vertex C of the triangle. All therefore that now remains to be done is to draw the perpendicular DC so that the solid $AB \times DC^2 = 2mD \times DC^2$ may be a maximum, or that $mD \times DC^2$ may be a maximum, which is obviously the same thing as within the given segment of the sphere to inscribe the greatest cylinder possible; and the method of doing this is too well known to need repeating here; it belongs to the Scholium to Theo. 19, Simpson on the Maxima and Minima.

XXVI. QUESTION 326, *answered by the* Proposer.

From the nature of *converging* series, it is evident that in the form proposed, x must necessarily be always less than *unity*. And

And it as evident that the coefficients, a, b, c, &c. must not only be in a regular decreasing progression, but in order that the terms of the series may continue *sine fine*, they must also be fractional numbers, or the reciprocals of an increasing progression of integers. Under these circumstances (for in no other can the *infinite* summation be possible) we may, by the assistance of NEWTON's residual theorem, always obtain an expression for the value of the proposed infinite series. Thus, if a, b, c, &c. be expounded by the reciprocals of the natural numbers 1, 2, 3, &c. the sum of the series resulting from the application of the theorem adverted to, will be had from this simple *formula*.—*Subtract* x *from unity, divide unity by the remainder, from the* n*th root of the quotient take unity, multiply the remainder by* n, *and divide the product by* x, *the quotient will give the sum of the series* ad infinitum; when n is any large power of 10. In like manner may general expressions be reduced from the said residual theorem with other values *ad libitum* of the coefficients a, b, c, &c. under the above limitations.

It is very remarkable that the above formula is obviously equal to *Napier's* logarithm of the reciprocal of $1 - x$ divided by x.

Hence, for example if x be $\frac{9}{10}$, the sum of the series will be expressed by——the log. of 10 divided by $\cdot 9$, equal to $2\cdot 558427$.

If x be $\frac{98}{99}$, the sum will be obtained by dividing the log. of 99 by the compound repetend $\cdot\dot{9}\dot{8}$, equal to $4\cdot 642$; and thus with any other values of x than unity.

Cavallerius refers the propoposer of this question to pa. 222, 223, &c. Simpson's Algebra, 2d edit. where he will find the method of summing the proposed series in a variety of cases, when the law of the coefficients is given.

XXVII. QUESTION 327, *answered by Mr*. Cunliffe.

In the present case $y = b$, and the equation of the curve becomes $x = \frac{l^2 - y^2}{2y} \times$ h. l. $(\frac{l + y}{l - y})$,

and

and taking the fluxions

$\dot{x} = \frac{l\dot{y}}{y} - \dot{y} \times (\frac{l^2+y^2}{2y^2}) \times$ h. l. $(\frac{l+y}{l-y})$, and hence

$y\dot{x} = l\dot{y} - \frac{\dot{y}}{2y} \times (l^2+y^2) \times$ h. l. $(\frac{l+y}{l-y})$

$= l\dot{y} - \frac{y\dot{y}}{2} \times$ h. l. $(\frac{l+y}{l-y}) - \frac{l^2\dot{y}}{2y} \times$ h. l. $(\frac{l+y}{l-y})$

= the fluxion of the area.

And fluent of $y\dot{x} = \left\{ \begin{array}{l} ly - \text{fluent of } \frac{y\dot{y}}{2} \times \text{h. l. } (\frac{l+y}{l-y}) \\ - \text{fluent of } \frac{l^2\dot{y}}{2y} \times \text{h. l. } (\frac{l+y}{l-y}). \end{array} \right.$

Now the fluent of $\frac{y\dot{y}}{2} \times$ h. l. $(\frac{l+y}{l-y})$ is

$= \frac{1}{2}ly - \frac{1}{4} \times (l^2 - y^2) \times$ h. l. $(\frac{l+y}{l-y})$

and the fluent of $\frac{l^2\dot{y}}{2y} \times$ h. l. $(\frac{l+y}{l-y})$ is

$= l^2 \times (\frac{y}{l} + \frac{y^3}{9l^3} + \frac{y^5}{25l^5} + \frac{y^7}{49l^7} + \frac{y^9}{81l^9} +$ &c.)

Whence the fluent of $y\dot{x}$ $\left\} = \left\{ \begin{array}{l} ly - \text{fluent of } \frac{y\dot{y}}{2} \times \text{h. l. } (\frac{l+y}{l-y}) \\ - \text{fluent of } \frac{l^2\dot{y}}{2y} \times \text{h. l. } (\frac{l+y}{l-y}) \end{array} \right.\right.$

$= \left\{ \begin{array}{l} \frac{1}{2}ly + \frac{1}{4} \times (l^2 - y^2) \times \text{h. l. } (\frac{l+y}{l-y}) \\ - l^2 \times (\frac{y}{l} + \frac{y^3}{9l^3} + \frac{y^5}{25l^5} + \frac{y^7}{49l^7} + \frac{y^9}{81l^9} + \text{\&c.}): \end{array} \right.$

but this expression ought to vanish when $y = l$, therefore the correct expression for the area is

=

$$=\begin{cases} l^2 \times (1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \&c.) \\ -\frac{l}{2} + \frac{ly}{2} + \frac{1}{4} \times (l^2 - y^2) \times \text{h. l.} (\frac{l+y}{l-y}) \\ - l^2 \times (\frac{y}{l} + \frac{y^3}{9l^3} + \frac{y^5}{25l^5} + \frac{y^7}{49l^7} + \frac{y^9}{81l^9} + \&c. \end{cases}$$

$$=\begin{cases} \frac{l^2q^2}{2} - \frac{l^2}{2} + \frac{ly}{2} + \frac{1}{4} \times (l^2 - y^2) \times \text{h. l.} (\frac{l+y}{l-y}) \\ - l^2 \times (\frac{y}{l} + \frac{y^3}{9l^3} + \frac{y^5}{25l^5} + \frac{y^7}{49l^7} + \frac{y^9}{81l^9} + \&c.) \end{cases}$$

becaufe $\frac{l^2q^2}{2} = l^2 \quad (1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81}$ &c.) as appears from Art. 6. of Landen's 5th Memoir; whence q denotes the length of the quadrantal arc of a circle whofe radius is 1. And when $y = 0$, or when the chain is completely ftretched along the horizontal line AB, the expreffion for the area will then be $\frac{l^2}{2} \times (q^2 - 1) = l^2 \times \cdot 7337005$.

The area may alfo be expreffed in other particular cafes of the value of y, without feries from what is deduced at Art. 40. Poftfcript to Landen's 5th Memoir.

The fame, by Mr. John Surtees.

Let $c = 3\cdot1416$, $z = AV =$ half the length of the chain, $x = SV$, and $y = AS = \frac{z^2 - x^2}{2x} \times \text{h. l.} (\frac{z+x}{z-x})$.

Then the fluxion of the area $= y\dot{x} =$

$\frac{z^2\dot{x}}{2x} \times \text{h. l.} (\frac{z+x}{z-x}) - \frac{x\dot{x}}{2} \times \text{h. l.} (\frac{z+x}{z-x})$.

The

The fluent of which is $\left\{\begin{array}{l} zx + \frac{x^3}{9z} + \frac{x^5}{25z^3} + \frac{x^7}{49z^5} \text{ \&c.} \\ + \frac{z^2 - x^2}{4} \times \text{h. l.} \left(\frac{z+x}{z-x}\right) - \frac{zx}{2}. \end{array}\right.$

And when $x = z$, the fluent is

$$= z^2 \times \left(1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49}\right) \text{\&c.} - \frac{z^2}{2} = z^2 \times \frac{c^2 - 4}{8}$$

$= z^2 \times \cdot 73369 =$ the area.

W. W. R.

XXVIII. QUESTION 328, *answered by Mr.* I. H. Hearding

Put $d = 25$ inches, the distance of the axis of motion from the centre of the ball, and $r = 5$ inches, the ball's radius; then, by Emerson's Mechanics, Prop. 58. the distance of the centre of oscillation from the point of suspension, or the true length of the pendulum, is $= d + \frac{2rr}{5d} = 25\frac{1}{5}$ inches. Let AC (fig. 668, pl. 32.) represent the wire when stretched out tight in an horizontal position, AB its position when it first becomes perpendicular to the same; and, from any point D, in the arch CBH, wherein the vibrations are made, draw $Dd \parallel CH$; and let $AB = AC$ be put $= a$, $Bd = x$, and the arch $BD = z$; then, by the nature of the circle, $\dot{z} = \frac{a\dot{x}}{\sqrt{(2ax - xx)}}$; and, since the fluxion of the time through BD is as $\frac{\dot{z}}{\sqrt{[Ad]}}$ (see Art. 207, Simpson's Fluxions) it will be here truly expressed by

$$\frac{a\dot{x}}{\sqrt{[a-x]} \times \sqrt{[2ax - x^2]}} = \frac{\sqrt{[\frac{1}{2}a]} \times \dot{x}}{\sqrt{[ax - xx]}} \times \left(1 - \frac{x}{2a}\right)^{-\frac{1}{2}}$$

$$= \frac{\sqrt{[\frac{1}{2}a]} \times \dot{x}}{\sqrt{[ax - xx]}} \times \left(1 + \frac{x}{2.2a} + \frac{3x^2}{2.4.4a^2} + \frac{3.5x^3}{2.4.6.8a^3} + \frac{3.5.7x^4}{2.4.6.8.16a^4}\right.$$
&c.)

and

and the fluent of this, when x becomes $= a$, is $=$

$$p\sqrt{[\tfrac{1}{2}a]} \times \left(1 + \frac{1}{2.2.2} + \frac{3.3}{2.2.4.4.2^2} + \frac{3.3.5.5}{2.2.4.4.6.6.2^3} \text{ \&c.}\right)$$

where p is $= 3{\cdot}14159$ &c. Now the time of descent through the diameter BF, or chord CB, is found to be as the fluent of

$\frac{2\dot{a}}{\sqrt{[2a]}}$, or $2\sqrt{[2a]}$; therefore, this is to the time of descent through the arc CDB,

$$\text{as } \frac{4}{p} : 1 + \frac{1}{2.2.2} + \frac{3.3}{2.2.4.4.2^2} + \frac{3.3.5.5}{2.2.4.4.6.6.2^3} + \text{\&c.}$$

$(= 1.180339)$.

But the absolute time of descent through the chord BC, or diameter BF is $= \sqrt{[\text{BF} \div 193]} = 0{\cdot}51237$ seconds (Simpson's Select Exercises, pa. 183, 1st. Edit.); consequently as $1{\cdot}27324 : 1{\cdot}180339 :: 0{\cdot}51237$ sec. : $0{\cdot}474986$ seconds, the time in which the wire will first become perpendicular to the horizon. *W. W. R.*

The same, by Mr. John Smith, *Alton Park.*

By Dr. Hutton's Dictionary, Vol. I. pa. 268, the distance of the centre of oscillation from the point of suspension will be $25 + \frac{2 \times 5^2}{5 \times 25} = 25{\cdot}4$. Then, by the nature of pendulums, as $\sqrt{39{\cdot}125} : \sqrt{25{\cdot}4} :: {\cdot}5'' : {\cdot}402865''$ the time of a semi-vibration in a cycloid. But by Dr. Hutton's Select Exercises, pa. 192, the time in a cycloid is to that in a circle, when the arc of semi-vibration is a quadrant. as 1 to $1{\cdot}18014$; therefore as $1 : 1{\cdot}18014 :: {\cdot}402865 : {\cdot}475437$ of a second, the time required.

The same, by Mr. John Surtees.

Let $r = \frac{1}{12} \times \left(25 + \frac{2 \times 5^2}{5 \times 25}\right) = \frac{127}{60}$ feet the distance of the centre of oscillation from the point of suspension, $c =$

$r\sqrt{[2]}$ = the chord of the arch which the centre of oscillation describes, $d = \frac{c}{4r^2}$, and $s = 16\frac{1}{12}$ feet.

Then $ca^2 = \frac{1}{2}$, and per Emerson's Fluxions, page 145, the time of a semi-vibration

$$= \frac{3\cdot1416}{4}\sqrt{[\frac{or}{s}]} \times : 1 + \frac{cd}{2^2} + \frac{3^2c^2d^2}{2^2\cdot4^2} \ \&c.$$

$$= \frac{3\cdot1416}{4}\sqrt{[\frac{254}{965}]} \times : 1 + \frac{1}{2^8} + \frac{1\cdot3^2}{2^4\cdot4^2} + \frac{1\cdot3^2\cdot5^2}{2^6\cdot4^2\cdot6^2} + \frac{1\cdot3^2\cdot5^2\cdot7^2}{2^8\cdot4^2\cdot6^2\cdot8^2} \ \&c.$$

$= \cdot477$ seconds.

W. W R.

XXIX. QUESTION 307, *answered by the* Proposer.

Let x denote the required length of the lever AB, w equal 2lb, its weight per inch, and p such a power applied at B, that may produce an equilibrium. Then from the principles of motion respecting the *heavy lever*, the momentum of the weight W, added to that of the lever itself, being equated to the momentum of the power acting at B, and properly transposed, it will then appear from the equation, that p is susceptible of a *minimum*. Therefore, by the fluxionary process *Maxima et Minima*, we find x equal to the square root of AC·W divided by half of w, equal to 60 inches, the required length of the lever, when the power applied at B to raise the weight and beam will be the *least possible*.

That is, $AC\cdot W + wx \times \frac{x}{2} = px$;

or, $\frac{AC\cdot W}{x} + \frac{wx}{2} = p = AC\cdot W \times x^{-1} + \frac{w}{2}x$;

hence $\frac{w}{2}\dot{x} = AC\cdot W \times x^{-2}\dot{x}$,

and $x = \sqrt{\frac{AC\cdot W}{\frac{1}{2}w}} = 60$.

The

The ſame reſult may be obtained without the aid of fluxions, from the conſideration that when there is an equilibrium of power and weight in any mechanical combination, the motive force then applied will be only what is neceſſary to overcome the *vis inertiæ* and the friction of the parts.

Hence, from the common laws of motion, if the weight at C be ſo applied (by means of a pulley) as to raiſe the other end of the lever vertically, the relation of the parts when there is an equilibrium, will be expreſſed by an equation, which reduced, will give AB, the length of the lever, equal to 60 inches, as before.

From the above ſolution we ſee the propriety of the obſervation made by the ingenious Mechanician M. De Prony, in his valuable treatiſe *Architecture Hydraulique*, Art. 298, reſpecting the *Heavy Lever*, which ignorance or prejudice has laboured ſo ſtrenuouſly to repreſent as abſurd,——*viz.*

"*Ainſi lorſqu'on voudra ſoulever une maſſe* W *avec un levier peſant* AB, *il y aura une certaine longueur à donner à ce levier, afin que le moteur qu'on y applique ſoit le moindre poſſible.*"

The ſame, by Mr. John Surtees, *Houghton-le-Spring, Durham.*

Let $z =$ the whole length of the lever in inches, then $2z$lbs. $=$ its weight, and of conſequence half that weight will be ſupported at B, and $\frac{36 \times 100}{z}$lbs. $=$ the part of the 100lbs. ſupported at B, which together $= z + \frac{3600}{z} =$ a minimum.

In fluxions $\dot{z} - \frac{3600\dot{z}}{z^2} = 0$. Hence $z = 60$ inches.

W. W. R.

The ſame, anſwered by Tyro Philomatheticus.

Let $x =$ the length required, $a = 36$ inches, and $w =$ weight of 100lbs. Then will $aw =$ force of the weight, and $2x \times$ ½

$px = x^2 =$ force of the lever; whence $\frac{aw + x^2}{x} =$ the power P, which, by the queſtion is to be a minimum.

In fluxions $\frac{2x^2\dot{x} - aw\dot{x} - x^2\dot{x}}{x^2} = 0$, or $x^2 = aw$.

Hence $x = \sqrt{(aw)} = 60$ inches to length of the lever.

The ſame, by Mr. James Cunliffe.

If we imagine the lever AB, (fig. 669, pl. 32.) to be ſupported by the fulcra DA and EB in a poſition parallel to the horizon the weight w of 100lb. being appended at C 36 inches from A; then the queſtion will be to determine the length of the lever AB ſo that the preſſure ſuſtained by the prop EB may be equal to the given weight p.

Put AC $= a$, and AB $= x$; then by the principles of mechanics, the part of the weight w ſuſtained by the prop EB will be expreſſed by $\frac{AC \times w}{AB} = \frac{aw}{x}$. Alſo the part of the weight of the leaver AB ſuſtained by the prop EB will be expreſſed by $\frac{AB \times s}{2} = \frac{sx}{2}$ where s denotes the weight of the lever per inch.

Therefore $\frac{aw}{x} + \frac{sx}{2} = \frac{2aw + sx^2}{2x}$ will expreſs the whole weight ſuſtained by the prop EB, which is to be equal to the given weight p by the queſtion,
that is $2aw + sx^2 = 2px$, and this equation properly reduced gives $x = \frac{p}{s} + \frac{\sqrt{(p^2 - 2saw)}}{s}$.

Whence it is evident that p^2 muſt not be leſs than $2saw$; and when $p^2 = 2saw = 14400$ then the weight ſuſtained by the prop EB will be the leaſt poſſible, and in that caſe $x = \frac{p}{s} = 60$ inches the length of the lever AB.

XXX. Or, PRIZE QUESTION 330, *answered by* Cavalerius.

Let BPC (fig. 700, pl. 33.) be the cycloidal curve, BO its axis, DC the base, BVD, MSC two semicircles (of the generating circle); and OS the line through the centres.

Draw EW, FX at equal distances from SO, and through K and L describe two semicircles, having their centres in SO. Then from the genesis of the curve, RP = PI; and because SP = PV, SR will be = IV, viz. CH = GD.

Torricellius and others have demonstrated that the trilineal spaces BMSCPB, BVDCPB are equal. Thus, because BM = CD, WK = LN, SP = PV, XL = KV, &c. each figure consists, or is made up, of an infinite number of parallel right lines, alike in each; and consequently the areas must be equal.

Now in a similar manner it is proved that the curve KPL bisects the mixt-lined quadrilateral KZLT; and that the trilineals KIP, LRP are equal; therefore the mixt-lined quadrilateral LRIT is = the trilinear LKT. Hence the required cycloidal space KLHG is equal to LRIT, = HG multiplied by the height DO; viz. equal to the distance of the centres of the circles multiplied by the radius.

This space may be cut out of the circle two ways. First, because ZK, LT are equally distant from SO, the arc KT, or ZL, is equal to LT or HG, therefore the cycloidal space is equal to the arc KT multiplied by the radius of the circle, or equal to a sector having its arc double KT. Secondly, if the distance of ZK and LT be made equal to the radius, the mixt-lined quadrilateral ZLTK will be equal to the cycloidal space, and may be cut out in that form.

The same, by the Proposer, Marloviensis.

Fig. 701, pl. 33. Let the cycloid be described as per quest. and the generating circle in the situation proposed, and also on the diameter BE, centre Z. At the intersection of the ordinates EK, FL, with the circumference of the generating circle, write

P, Q; and join ZP, ZQ. Draw alſo the chords BP, BQ, DP, DQ, GK, HL. Then, from the well known properties of the circle and cycloid, we have

$$\left.\begin{matrix}\text{the cycloidal}\\ \text{ſpace BLHD}\end{matrix}\right\} = \left\{\begin{matrix}\text{BF} \times \text{FL} - \text{circ. area BQF} + \text{rectilineal}\\ \text{ſpace FLHD} + \text{circ. ſegm. HL.}\end{matrix}\right.$$

And in like manner

$$\left.\begin{matrix}\text{the cycloidal}\\ \text{ſpace BKGD is}\end{matrix}\right\} = \left\{\begin{matrix}\text{BE} \times \text{EK} - \text{circ. area BPF} +\\ \text{rectilin. ſpace EKGD} + \text{circ. ſeg. GK};\end{matrix}\right.$$

the difference of which will evidently exhibit an expreſſion for the area of the cycloidal ſpace KLHG required. This method is general, wherever the points E, F are taken; but, when they are at equal diſtances from the centre Z, as in the queſtion, it becomes a particular caſe only, and the above general expreſſion reduces by expoſition to a very remarkable ſimple concluſion.

From the *Data*, and the properties of the figure, we have
$BZ = ZD = ZP$, $EZ = ZF$, $EP = FQ$,
arc $BP =$ arc $DQ = DG = PK$, arc $PQ = GH$,
arc $BQ =$ arc $DP =$ arc $BP +$ arc PQ,
$BE = DF = BZ - EZ$, $BF = BZ + EZ$,
$EK = EP\ (FQ) + PK\ (DG) = EP +$ arc BP (arc DQ),
$FL = EP\ (FQ) +$ arc BP (arc DQ) $+$ arc PQ.

Call the area of the generating circle, A; the circular area DQF, or BPE, X; and the circ. ſegm. on HL, BP, or DQ, Y; and, for brevity, the propoſed cycloidal ſpace, S: And the circ. ſegm. on DP, BQ, or GK will be denoted by
$\frac{1}{4}A -$ triang. $BDQ - Y$, or $\frac{1}{4}A - BZ \times EP - Y$;
the circ. area BPQF by $\frac{1}{4}A - X$; and the rectilineal ſpaces FLHD, EKGD, reſpectively by

$$\frac{FQ}{2} \times FD + FD \times \overline{BP + \text{arc } PQ} =$$

$$\overline{BZ - EZ} \times \overline{\frac{FQ}{2} + BP + \text{arc } PQ}; \text{ and}$$

$$\frac{FQ}{2} \times \overline{BE + EZ} + \text{arc } BP \times BF.$$

Hence the former of the above expreſſions becomes

$$\overline{BZ + EZ} \times \overline{EP + \text{arc } BP + \text{arc } PQ} - \frac{A}{2} + X$$

$+ \overline{BZ - EZ} \times \overline{\tfrac{1}{2}FQ + \text{arc } BP + \text{arc } PQ}$
$+ Y$; and the latter

$\overline{BZ - EZ} \times \overline{FQ + \text{arc } BP} - X$
$+ \overline{\tfrac{1}{2}FQ + \text{arc } BP \, (\text{arc } DQ)} \times \overline{BZ + EZ}$

$$+ \frac{A}{2} - \overline{BZ \times FQ} - Y;$$

which being expanded by multiplication, and reduced, become respectively.

$\frac{3}{2}BZ \times EP + 2BZ \times \text{arc } BP + \tfrac{1}{2}EZ \times EP - \frac{A}{2} + X$

$+ 2BZ \times \text{arc } PQ + Y =$ the cycloidal space BLHD;
and $\frac{3}{2}BZ \times EP + 2BZ \times \text{arc } BP - \tfrac{1}{2}EZ \times EP$

$+ \frac{A}{2} - X - Y =$ the cycloidal space BKGD.

The latter subtracted from the former, leaves
$BZ \times EP + EZ \times EP + 2X + 2Y + 2BZ \times \text{arc } PQ - A = S$.
But $BZ \cdot EP + 2Y$ is $=$ twice the sector BZP,
so is $EZ \cdot EP + 2X =$ twice the same sector;
and the quantity $2BZ \times$ arc PQ is evidently $= 4$ times the sector ZPQ; therefore
$4 \times$ sector $BZP + 4 \times$ sector $ZPQ - A = S$;
or, because 4 sector $BZQ = A + 2$ sector PZQ,
twice the sector PZQ is $=$ the cycloidal space KLHG.

Hence, since GH is equal the arc PQ, the concise theorem required is——*The product of the semidiameter of the generating circle into the distance* GH *is equal the cycloidal space contained within the arcs* KG *and* LH; and the cutting off a space from the generating circle equal to the cycloidal space *geometrically* is obvious, being the circular sector on twice the arc PQ.

Q. E. O.

The same, by Mr. Wallace, *of the Royal Military College.*

Fig. 702, pl. 33. Instead of supposing the points E and F equally distant from the centre, let us suppose them taken any where

where in the axis. Let EK, FL, meet the generating circle in M and N. Draw the chords DM, DN, GK, HL; then from the nature of the curve DM, GK are parallel, alſo DN, HL. Now it was firſt ſhewn by Dr. Wallis (in his Treatiſe *De Motu* Cap. V. Prop. 20.) that if FL be any ordinate to the axis, meeting the generating circle in N, and LH be drawn parallel to the chord DN, meeting the baſe in H, the cycloidal area BKLHD is triple the area contained by the arch BMN, the axis BD, and the chord DN, to theſe let the equal ſegments HpL, DPN of the generating circle be added, and the mixtilineal area BKLpHD will be equal to half the generating circle together with twice the trilineal area BMND. For the ſame reaſon the mixtilineal area BKqGD is equal to half the generating circle together with the trilineal area BMD, therefore, taking the latter equation from the former, the propoſed cycloidal ſpace LpHGqK is double the area contained by the chords DM, DN and the arch MN. Hence the Theorem, as well as the geometrical method required in the queſtion is obvious, not only for that particular caſe, but for any given poſitions of the generating circle whatever.

Ingenious ſolutions to this queſtion were alſo received from Meſſrs. Cunliffe, Surtees, *and* Whitley.

The Medal for ſolving the Mathematical Prize Queſtion is decided in favour of Mr. JOHN SURTEES, who will pleaſe to ſend for it to Mr. GLENDINNING's.

ARTICLE XXII.

On the Quadrature of the Circle.

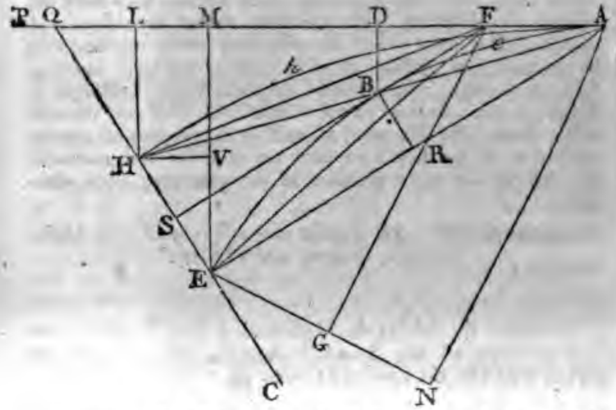
By Mr. Benjamin Gompertz, *London.*

A certain Author attempting to ſquare the circle expreſſes himſelf to the following effect.

"If a tangent be drawn to a given circle, and from the point of contact a line be drawn through the centre and produced indefinitely on that ſide of the tangent where the centre of the circle is, and

and there be taken an indefinite number of points in the lines thus produced beyond the centre and arcs of circles be described about every one of these as centres, through the said point of contact on one and the same side of the said line produced and each equal in length to the quadrant arc of the given circle reckoning from the aforesaid point of contact to points in the circular arcs, now described, that the locus of these points in the arcs will be a circle, and moreover that the centre thereof will be in the said tangent."

In order to determine the truth or fallacy of this I will premise the following Lemma.



LEMMA. Let PA be the aforesaid tangent, A the aforesaid point of contact, AH, AE the two lines passing through A and cutting the locus in H and E, the ∠ EAP being double HAP, or which is the same thing HAE = HAP, also let A*h*H, A*e*E be arcs of circles of equal length (and equal to the quadrantal arc of the given circle) passing through H, A, and E, A whose centres are on the same side of AP and in the line perpendicular thereto at A; join H, E and make HL perpendicular to AP; then I say HE is equal to HL and angle HEA a right angle.

For from the point B, the intersection of A*e*E and AH, draw BD and BR perpendicular to AP and AE, and join B, E; now since the centres of A*h*H and A*e*B are in the line perpendicular to AP at A, AP is a tangent to both, consequently the angle HAL, the angle of chord and tangent of both arcs, being the same

ſame the arcs are ſimilar. Furthermore the angle AEB in the alternate ſegment is equal to the angle BAD, which by hypotheſis is equal the angle BAE, therefore BAE = BEA, therefore AR = AD = RE, conſequently 2AD = AE, and AeE, that is by hypotheſis, AhH = 2AeB, therefore is AH = 2AB, or by ſimilar △s AL = 2AD, equal from above to AE; hence the triangles AHL and AHE having AH common, the angles at A equal, and the ſide AL in the one equal to AE in the other, are equal in every reſpect; therefore HE = HL and ∠ HEA equal the right ∠ ALH. *Q. E. D.*

Corollary. The tangent to the locus at its interſection with AP is perpendicular to AP, and conſequently if the figure were a circle the centre would be in the line AP and therefore the latter part of the aſſertion would be true; for produce HE both ways to C and to Q, its interſection with AP, then the ∠ CQA is equal a right angle leſs the angle EAQ, but it is evident if H and E were to conicide CQ would be a tangent to the coincident points H and E, and then the angle EAH vaniſhing, EAQ its double muſt alſo vaniſh, and therefore E, H, Q coincide with the interſection of the locus and AP; alſo Q a right angle and CH a tangent to the locus at the ſaid interſection. *Q. E. D.*

Hence the locus cannot be a circle. For draw the right line AN making the angle NAP equal to 2EAP cutting the locus in N; join EN, biſect HE and EN in S and G; draw SF perpendicular to HE; join H, E, and G with F, and draw ME perpendicular and AP parallel to AP and ſuppoſe the locus a circle then its centre muſt be in the line SF biſecting the chord at right angles, but by the Corollary to the Lemma it is in AP, therefore F is the centre; conſequently the angle G of a line from the centre biſecting a chord a right angle, but by the Lemma ANE and AEH are right angles, therefore FS and FG parallel to AE and AN reſpectively; but by conſtruction, the angle NAP is double the angle EAP conſequently, ſubſtituting equals for equals, the angle GFP is double the angle SFP or the angle GFS equal to SFP, but SFE is equal to HFS, conſequently taking equals from equals, the angles EFG and HFP are equal, conſequently the triangles EFG and HFL having the angles at F equal, the angles at L and G equal, being both right angles, and the ſides HF and EF, being the radii of the ſame circle, equal; they are equal in every reſpect, and conſequently HL and EG equal, conſequently 2EG or EN and 2HL or HL more MV are equal, but by the Lemma EN is equal to EM or VE more MV, and HL equal to HE, conſequently VE and MV is equal to HE and MV,

and

and confequently VE and HE are equal, that is, the hypothen
of a right angled triangle equal to one of the other fides whic
abfurd. *Q. E. I*

Note. Let γ denote the radius of the given circle, q
length of a quadrant in the circle having unity for radius, and
the radius of one of the circles touching the given line at the gi
point of contact; then $\frac{\gamma}{x} \times q$ will denote the meafure of the a
intercepted between the point of contact and the locus; and
length of the chord of the arch will be $2x \times \text{fin.}\ \frac{\gamma}{2x} \times q$.

But the angle made by the chord with the given tangent w
be half the angle at the centre ftanding upon the arch; that i
will be meafured by $\frac{\gamma}{2x} \times q$. Hence if we conceive the lo
to be generated by the gyration of a ftraight line about the gi
point of contact as a pole, and call ρ the revolving radius and φ
meafure of the angle made by it with the given tangent, we fh
have

$$\rho = 2x \times \text{fin.}\frac{\gamma}{2x} \times q.$$

$$\varphi = \frac{\gamma}{2x} \times q.$$

And exterminating x, the equation of the curve is

$$\rho = \gamma \times \frac{q}{\varphi}\ \text{fin.}\ \varphi.$$

It may be remarked that by taking x very great, or φ very fma
the curve approaches to a circle of which $\gamma \times q$ is the radius, a
the point of contact the centre, and that it ultimately coinci
with this circle when x is infinite.

THE END.

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